MAT 319: HOMEWORK 4

DUE TUESDAY, JANUARY 20

In problems involving real numbers, please make sure that you are only using axioms of real numbers and results proved in the book. Give precise references, e.g. "by Theorem 2.1.7", "by Axiom M4".

1. If $a, b \in \mathbb{R}$, prove that
   (a) $(-1)a = -a$;
   (b) $1/(-a) = -(1/a)$.

2. If $a \in \mathbb{R}$ satisfies $a \cdot a = a$, then $a = 0$ or $a = 1$.

3. Show that the equation $x^2 = 3$ has no solutions for $x \in \mathbb{Q}$. (That is, $\sqrt{3}$ is irrational.)

4. Let $a, b \in \mathbb{R}$ be such that $0 < a < b$. Prove by induction that for any positive integer n, $a^n < b^n$.

5. Let $a, b \in \mathbb{R}$. Show that $a^2 + b^2 = 0$ if and only if $a = 0$ and $b = 0$. [Hint: use order relation.]

6. Prove by induction that $|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$ for all $a_1, a_2, \ldots a_n \in R$.

7. Let $a, b, x, y \in \mathbb{R}$ be such that $a < x < b$, $a < y < b$. Show that then $|x - y| < b - a$.

8. Let $a, b \in \mathbb{R}$, $a \neq b$. Show that there exist ε-neighborhoods of a, b which do not intersect: $V_\varepsilon(a) \cap V_\varepsilon(b) = \varnothing$.