MAT 319: HOMEWORK 2

DUE TUESDAY, FEB. 6

1. Prove that for any positive integer n,

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{4n^3 - n}{3}$$

2. Find all n such that $n^2 < 2^n$. Prove your answer by induction.

3. Guess a general formula for the product

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\cdots\left(1 - \frac{1}{n^2}\right)$$

and prove it by induction.

4. Let the Fibonacci sequence be defined by $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$ for $n > 1$. Prove that

(a) $F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}$,

(b) $F_2 + F_4 + \ldots F_{2n} = F_{2n+1} - 1$.

5. Suppose you are given 32 cups, each containing some water (possibly different amount in different cups). You would like to make all cups hold the same amount of water by pouring water from some cups into others. Indeed, you are allowed to choose any two cups and pour some water from one cup to the other to equalize the amount of water in these two cups. Show that you can make *all* cups to hold the same amount of water by performing a finite sequence of such moves. [Hint: prove the statement for 2^n cups by induction.]

6. Let numbers x_n be defined recursively by $x_1 = 1$, $x_2 = 2$, $x_3 = \frac{1}{2}$, and

$$x_n = \frac{x_{n-1} + x_{n-2} + x_{n-3}}{3}, \quad n > 3.$$

Show that $0 \le x_n \le 2$ for all n. (Use strong induction.)