MAT 319: HOMEWORK 11

DUE TUESDAY, APRIL 24

1. For each of the following functions, find out whether the limit exists. If it does, find it. (You may use the limit theorems, but you have to explain exactly how you apply them.)
(a) $\lim_{x\to 3} \frac{x^2-9}{x^3-27}$,
(b) $\lim_{x\to 0} x^2 \cos \frac{1}{x}$,
(c) $\lim_{x\to 1} \frac{1}{1-x^2}$.

2. Let $f(x)$ be the function defined by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational.} \end{cases}$$

Do the following limits exist? Prove your answer.
(a) $\lim_{x\to 1} f(x)$;
(b) $\lim_{x\to 2} f(x)$.

3. **(Sequential Criterion for Continuity)** Show that a function f is continuous at $x = c$ if and only if for every sequence (x_n) converging to c the sequence $f(x_n)$ converges to $f(c)$. (This is similar to the Sequential Criterion for Limits that we proved in class. Be careful: in that case we only allowed sequences with $x_n \neq c$.)

4. Use Sequential Criterion to show that $\lim_{x\to 0} \cos \frac{1}{x}$ does not exist. **Hint:** find sequences (x_n) and (y_n) converging to 0, such that $\cos \frac{1}{x_n} = 1$ and $\cos \frac{1}{y_n} = -1$.

5. Suppose that functions $f(x)$, $g(x)$ are defined for all x.
(a) Show that if both $\lim_{x\to c} f$ and $\lim_{x\to c}(f + g)$ exist, then $\lim_{x\to c} g$ exists.
(b) If $\lim_{x\to c} f$ and $\lim_{x\to c} fg$ exist, does it follow that $\lim_{x\to c} g$?