MAT 319: HOMEWORK 10

DUE TUESDAY, APR 17

1. Suppose that a sequence (x_n) diverges to $+\infty$.
  (a) Show that (x_n) has an increasing subsequence. (Do not use Theorem 3.4.7; rather, give a direct proof.)
  (b) Prove that (x_n) cannot have a decreasing subsequence.

2. (a) Find a $\delta > 0$ such that $|x^2 - 9| < \frac{1}{10}$ whenever $|x - 3| < \delta$. (You don't have to find the best possible δ, but you have to show that your δ works.)
  (b) Using the ϵ-δ definition, show that the function $f(x) = x^2$ is continuous at $x = 3$.

3. Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} x & \text{if } x > 0, \\ 3 & \text{if } x \leq 0. \end{cases}$$

  (a) Using the ϵ-δ definition of the limit, show that $\lim_{x\to 0} f(x) \neq 3$.
  (b) Show that $\lim_{x\to 0} f(x)$ does not exist.
  (Please do this question from scratch; do not use one-sided limits.)

4. Suppose that $\lim_{x\to c} f(x) = L$. Let $g(x) = 5f(x)$. Using the ϵ-δ definition of the limit, show that $\lim_{x\to c} g(x) = 5L$.

5. **(One-sided limits)** Here's the definition.
We say that $\lim_{x\to c+} f(x) = L$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$0 < x - c < \delta \Longrightarrow |f(x) - L| < \epsilon.$$

Similarly, $\lim_{x\to c-} f(x) = L$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$0 < c - x < \delta \Longrightarrow |f(x) - L| < \epsilon.$$

Using these definitions, show that $\lim f(x) = L$ if and only if $\lim_{x\to c+} f(x) = \lim_{x\to c-} f(x) = L$. (More precisely, the limit exists and equals L if and only if both one-sided limits exist are equal L.) This is Theorem 4.3.3 in the book, but the book gives no proof.