MAT 319: HOMEWORK 1

DUE TUESDAY, JAN. 30

1. Prove the identity: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

2. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 - x$.

(a) Graph it.
(b) Find $f(A)$ where A is the open interval $(0, 4)$.
(c) Find $f^{-1}(B)$ where $B = [1, 4]$.
(**Note:** as we discussed in class, $f^{-1}(B)$ stands for the *preimage* of B, and makes sense even though f is not bijective, and so the inverse function f^{-1} cannot be defined.)
(d) Find two subsets C, D of $\mathbb{R}$ such that $f(C) \cap f(D) \neq f(C \cap D)$.

3. Let $f\colon A \to B$ be a function and suppose that $C \subseteq A$ and $D \subseteq B$. Are the following statements true or false (for every choice of f, C, D)? Justify your answers by a brief proof or a counterexample.

(a) $f(A \setminus C) \subseteq f(A) \setminus f(C)$.
(b) $f^{-1}(B \setminus D) = f^{-1}(B) \setminus f^{-1}(D)$.

Hint: as in question 2, try some examples. You can try functions $f\colon \mathbb{R} \to \mathbb{R}$ or you can try functions $f\colon A \to B$ where A and B are finite sets.

4. Suppose that $f\colon A \to B$ and $g\colon B \to C$ are functions such that the composition $g \circ f$ is injective. Is f necessarily injective? What about g? Give brief proofs or counterexamples.

5. Let A, B be two subsets of $\mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ be a function. Construct negations of the following statements (without using the words "it is not true", "there does not exist", etc). Explain how you did it. You don't have to worry about the validity of the statements themselves.

(a) For every $x \in A$ and every $y \in B$ $f(x) < f(y)$.
(b) There exists a natural number $n \in \mathbb{N}$ such that $x < n$ for all $x \in A$.
(c) There exists a natural number $n \in \mathbb{N}$ such that for all $x \in \mathbb{R}$

$$x > n \Longrightarrow f(x) > 1.$$

(d) For every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in \mathbb{R}$

$$|x - y| < \delta \Longrightarrow |f(x) - f(y)| < \epsilon.$$