MAT 319 Introduction to Analysis

Practice Questions for Final

The questions below are meant to give you some extra practice for (some of) the material we covered this semester. They are not supposed to mimic an actual exam, nor do they provide a comprehensive coverage for everything we studied. Some of these questions are harder than the exam questions would be.

There will be 10 questions on the final exam (worth 10 points each).

1. Let $g_1, g_2 : A \to B$, $f : B \to C$ be functions such that $f \circ g_1 = f \circ g_2$.
  (a) Prove that if f injective, then $g_1 = g_2$.
  (b) Show that if f is not injective, then it is possible that $g_1 \neq g_2$.

2. Prove that $1^3 + 2^3 + \dots n^3 = (1 + 2 + \cdots + n)^2$ for all natural n.

3. Consider the set $S = \{a\sqrt{2} + b\sqrt{3} : a, b\ in \mathbb{Z}\}$. Is this set countable? Prove your answer.

4. Let $f : A \to \mathbb{R}$ be any function, and let $u = \inf\{f(x) : x \in A\}$. Show that there is a sequence $x_n \in A$ such that $\lim f(x_n) = u$.

5. Suppose $\lim x_n = 4$. Prove that $\lim \frac{x_n}{x_n - 2} = 2$ in two ways:
  (a) arguing from definition;
  (b) using the limit laws.

6. Let (x_n) and (y_n) be two convergent sequences. Define the sequence (z_n) by $z_n = \max\{x_n, y_n\}$. Prove that the sequence (z_n) converges. **Hint:** consider two cases, 1) $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$, and 2) $\lim_{n\to\infty} x_n \neq \lim_{n\to\infty} y_n$.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$|f(x)| \leq 3\sqrt{|x - 1|}.$$

Using the ϵ-δ definition of a limit, prove that $\lim_{x\to 1} f(x) = 0$. Is f continuous at $x = 1$?

8. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at $p \in \mathbb{R}$. If $f(p) > A$, show that there is a $\delta > 0$ such that $f(x) > A$ whenever $|x - p| < \delta$.

9. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that f is discontinuous at every point, but $|f|$ is continuous at every point.

10. If $f : [0, 1] \to \mathbb{R}$ is continuous and has only rational values, must f be constant? Explain.