MAT 319 projects

1. **Cantor's Theorem.** We saw that the set $\mathbb{R}$ is "larger" than $\mathbb{N}$ (that is, $\mathbb{R}$ is uncountable, and there is no bijection between $\mathbb{N}$ and $\mathbb{R}$). Is there a set which is even larger than $\mathbb{R}$? The answer is yes. In fact, Cantor's theorem says that the set of all subsets of A is always "larger" than A.
2. **Decimal presentations for rational and irrational numbers.** There is a nice way to tell rational numbers from irrational by looking at their decimal presentation.
3. **Transcendental numbers.** A number is *transcendental* if it is not a root of any polynomial with integer coefficients. In particular, all transcendental numbers are irrational. (Why?) They are quite hard to construct explicitly, but you can show that transcendental numbers form an uncountable, dense subset of $\mathbb{R}$. For a more ambitious project, you can try to understand how transcendental numbers can be constructed (this is related to *Liouville numbers*), and even prove that e is transcendental.
4. **The Cantor set.** This is a subset of $[0, 1]$ which can be defined in terms of ternary (base 3) presentation of real numbers. This set has a number of interesting properties, and is related to "fractals". To construct this set, you remove from $[0, 1]$ a collection of intervals whose total length is 1; however, what remains is non-empty (and even uncountable).
5. **Euler's number e.** There are a few different definitions of the number e. You can find out what they are, and why these definitions are equivalent. You can also prove that e is irrational. You could also find out how e is related to banking and compound interest.
6. **Cauchy sequences** An important Cauchy Criterion allows to determine whether the sequence is convergent or not – you don't even have to guess the limit! The idea is that the terms of a convergent sequence must get closer and closer to one another.
7. **Accumulation points.** If a sequence (x_n) is divergent, its different subsequences may converge to different limits. Each of those limits is called an *accumulation point* for (x_n). You can also talk about accumulation points of a subset of $\mathbb{R}$. For example, the set of accumulation points of $\mathbb{Q}$ is $\mathbb{R}$. This implies that the set of all subsequences of $\mathbb{Q}$ is uncountable.
8. **A "wildly divergent" sequence.** Consider the sequence (x_n), $x_n = \sin n$. This sequence has a subsequence converging to any given number $A \in [-1, 1]$.
9. **The Fibonacci sequence.** This is the well known sequence 1, 1, 2, 3, 5, 8, ... defined inductively by the equation $F_{n+1} = F_n + F_{n-1}$. It turns out that the ratios of successive terms $r_n = F_{n+1}/F_n$ form a sequence that is eventually monotone and converges to the number

known as the Golden Ratio Φ (the positive root of the equation $x^2 = x + 1$.) Explore the effect of starting with pairs of different numbers, eg with 1, 5 or with -2, 1. Do these sequences also converge? If so, what are their limits?

10. **Open and closed sets.** These give you a glimpse into topology. You can define open sets via ϵ-neighborhoods and closed sets as their complements, or you can define closed sets via accumulation points. The simplest examples are open and closed intervals.
11. **Infinite series.** In calculus, you've probably seen infinite sums called series. Now with a precise definition of a limit, you can describe what exactly is meant by an infinite sum, and give examples of convergent and divergent series.
12. **Infinite series and the Riemann theorem.** Finite sums are independent of the order of summands. Surprisingly, this is not true for infinite sums. A beautiful theorem of Riemann says if a series is "conditionally convergent", you can rearrange its terms to obtain any given number as the sum of the series!
13. **Euler's product.** Give an accurate definition of infinite product $\prod_{n=1}^{\infty} a_n$ as a limit. Describe the relation between convergence of the sum $\sum a_n$ and product $\prod(1+a_n)$. Accurately prove Euler's formula:

$$\prod_{i=1}^{\infty} \frac{1}{1 - p_i^{-s}} = \sum_{n=1}^{\infty} n^{-s}, \qquad s > 1$$

where $p_1 = 2$, $p_2 = 3$, $p_3 = 5 \dots$ is the sequence of all prime numbers.

14. **Monotone functions.** These have some interesting properties. Theorem: a monotone function can be discontinuous only at countably many points.
15. **Metric spaces.** The notions of bounded sets, limits of sequences, etc. can be defined not only for intervals (or other subsets of $\mathbb{R}$), but for some more general spaces (provided we have a "distance" between any two points of the space).
16. **Uniform Continuity.** A function is continuous at a point x_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$. If the fucntion is continuous at every point, the choice of δ (for a given ε) may vary with x_0. A function is *uniformly continuous* if one choice of δ works for all points. Give examples of continuous functions that are not uniformly continuous. Explain why any continuous function defined on the interval $[a, b]$ is uniformly continuous.