MIDTERM II SOLUTIONS

1.

(a) A function $f : E \to \mathbb{R}$ is bounded if there exists some constant $M \in \mathbb{R}$ such that $|f(x)| < M$ for all $x \in E$.

(b) The function $f(x) = 1/x$ is unbounded on the domain $(0, 1)$.

(c) Suppose for the sake of a contradiction that f is unbounded. Then for each $n \in \mathbb{N}$, we can find some $x_n \in [a, b]$ such that $|f(x_n)| > n$. By the Bolzano-Weierstrass theorem, (x_n) contains a convergent subsequence (x_{n_k}) with limit $x_0 \in [a, b]$. On one hand, by continuity of f we have

$$\lim_{k\to\infty} f(x_{n_k}) = f(x_0).$$

On the other hand, by construction $|f(x_{n_k})| > n_k$, and therefore $\lim_{k\to\infty} f(x_{n_k})$ cannot exist (as a finite value). This is a contradiction, so f must be bounded.

2.

(a) (i) If $\lim_{n\to\infty} x_n = -\infty$, then $\lim_{n\to\infty} f(x_n) = +\infty$.

(ii) For every $M > 0$, there exists some $N < 0$ such that $f(x) > M$ for all $x < N$.

(b) (ii) $\Rightarrow$ (i) Let (x_n) be a sequence that diverges to $-\infty$. Fix some $M > 0$, and let $N < 0$ be as given in definition (ii). Then there exists some K such that $x_n < N$ for all $n > K$. Thus $f(x_n) > M$ for all $n > K$, proving that $\lim_{n\to\infty} f(x_n) = +\infty$. This establishes (i).

(i) $\Rightarrow$ (ii) We will prove the contrapositive. Suppose that (ii) does not hold. Then for some $M > 0$, the statement

$$x < N \Rightarrow f(x) > M$$

fails for all $N < 0$. This means for each $n \in \mathbb{N}$, there exists some $x_n < -n$ such that $f(x_n) \leq M$. Then the sequence (x_n) diverges to $-\infty$, but $\lim_{n\to\infty} f(x_n)$ does not equal $+\infty$ (and may not even exist).

3.

(a) Consider the sequence defined by $x_n = 1/(n+1/2)$. This sequence converges to 0, but

$$g(x_n) = \sin\left(\frac{\pi}{1/(n+1/2)}\right) = \sin(n\pi + \pi/2) = (-1)^n,$$

and therefore $\lim_{n\to\infty} g(x_n)$ does not exist. This proves that $\lim_{x\to 0} g(x)$ does not exist.

(b) Observe that $\sin(x)$ is continuous at all x and π/x is continuous at all nonzero x. Therefore by the composition law, $\sin(\pi/x)$ is continuous at all nonzero x and in particular, $x = 1$. Then $g(x)$ is continuous at $x = 1$, so

$$\lim_{x\to 1^-} g(x) = g(1) = \sin(\pi) = 0.$$

(c) We restrict our attention to x in the interval $(-1, 0)$. Observe that

$$\frac{\sqrt{x+1}-1}{x} = \frac{\sqrt{x+1}-1}{x} \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} = \frac{1}{\sqrt{x+1}+1}$$

for all $x \neq 0$, and this new expression is continuous at zero. (It is a composition of continuous functions, and the denominator is nonzero at $x = 0$.) Therefore

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \frac{\sqrt{x+1}-1}{x} = \lim_{x\to 0^-} \frac{1}{\sqrt{x+1}+1} = \frac{1}{2}.$$

4.

(a) Since $a_n/n \leq a_n$ for all $n \geq 1$, it follows from the comparison test that

$$\sum_{n=1}^{\infty} \frac{a_n}{n} \leq \sum_{n=1}^{\infty} a_n < \infty.$$

(b) Consider $a_n = 1/n$. Then $\sum_{n=1}^{\infty} a_n$ is the harmonic series which diverges, but $\sum_{n=1}^{\infty} a_n/n = \sum_{n=1}^{\infty} n^{-2}$ which converges by the p-test ($p = 2 > 1$).

(c) We want to apply the alternating series test, which says that if b_n is a nonnegative and decreasing sequence that converges to 0, then $\sum_{n=1}^{\infty}(-1)^n b_n$ converges. Thus, it will suffice to show that $a_n/\sqrt{n}$ is nonnegative, is decreasing, and converges to 0. Nonnegativity is immediate, since $a_n \geq 0$. Also, $a_{n+1} \leq a_n$ and $\sqrt{n+1} > \sqrt{n}$, so

$$\frac{a_{n+1}}{\sqrt{n+1}} \leq \frac{a_n}{\sqrt{n}}.$$

This shows that $a_n/\sqrt{n}$ is a decreasing sequence. Now let $\epsilon > 0$ be given, and choose $N = (a_1/\epsilon)^2$. Then if $n > N$,

$$\begin{aligned} n &> \left(\frac{a_1}{\epsilon}\right)^2 \\ \sqrt{n} &> \frac{a_1}{\epsilon} \\ \epsilon &> \frac{a_1}{\sqrt{n}}. \end{aligned}$$

Since a_n is a decreasing sequence,

$$\frac{a_n}{\sqrt{n}} \leq \frac{a_1}{\sqrt{n}} < \epsilon$$

for all $n > N$. This proves that $a_n/\sqrt{n}$ converges to 0, so we are done.