MAT 319
Proof of the intermediate value theorem

We will give a proof that is slightly different from the one in the book, in particular, it uses ϵ-δ-approach rathen than sequences. (Please read the proof in the book, it's also a good proof!)

We will need the following lemma (a version was proved in class, another version is on the homework). We don not include a proof here.

Lemma 1. *(1) Suppose f is a function continuous at a point z, and $f(z) > c$. Then there is $\delta > 0$ such that for every $x \in (z - \delta, z + \delta)$, we have $f(x) > c$ (as long as $x \in dom(f)$).*

(2) Suppose f is a function continuous at a point z, and $f(z) < c$. Then there is $\delta > 0$ such that for every $x \in (z - \delta, z + \delta)$, we have $f(x) < c$ (as long as $x \in dom(f)$).

Now we prove the intermediate value theorem: suppose f is continuous on $[a, b]$, $f(a) < c$, $f(b) > c$. We need to show that there is a point $x_0 \in (a, b)$ such that $f(x_0) = c$.

Since $f(a) < c$, Lemma 1 implies that $f(x)$ stays less than c for x close to a. Let's travel from a towards b, and see how far we can get while the values of f stay less than c. To make this precise, consider the set

$$S = \{x \in [a, b] : f(x) < c, \text{ and the values of } f \text{ are less than } c \text{ at all points between } a \text{ and } x\}.$$

You can actually show that the set S is just an interval starting at a. Importantly for us, S is non-empty (because it contains a), and S is bounded, because $S \subset [a, b]$. By Completeness Axiom, S has a supremum.

Consider $x_0 = \sup S$. We will show that $f(x_0) = c$. Indeed, we will rule out the possibilities $f(x_0) < c$ and $f(x_0) > c$; this will mean $f(x_0) = c$.

First, let's assume $f(x_0) > c$. Then by Lemma 1, there is $\delta > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta)$ we have $f(x) > c$. This means (why?) that the interval $(x_0 - \delta, x_0 + \delta)$ contains no points of S. But ths contradicts (why?) the fact that $x_0 = \sup S$.

Now, let's assume that $f(x_0) < c$. In this case, we will show that the set S extends to the right of x_0, so x_0 cannot be an upper bound for S. This will again give a contradiction. Indeed: by Lemma 1, there is $\delta > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta)$ we have $f(x) < c$. Now, since $x_0 = \sup S$, there must (why?) be a point $x' \in S$ such that $x_0 - \delta < x' \leq x_0$. But now we have that $f(x) < c$ for all points between a and x', including x' (why?), and then $f(x) < c$ for all points between x' and $x_0 + \delta$. But this means (why?) that the set S contains points x with $x > x_0$, a contradiction with $x_0 = \sup S$. $\square$

Please make sure you can answer all the "why?". Make a picture to understand this proof better.