HOMEWORK 7 SOLUTIONS

15.2

(a) The series diverges. Observe that for $n = 6k+3$, $[\sin(n\pi/6)]^n = 1$. Since the terms of the series do not converge to zero, the sum cannot converge.

(b) The series converges. Since $n/7$ is never a half integer, $\sin(n\pi/7) < 1$ for all n. In fact, $\sin(n\pi/7)$ only takes on finitely many values, so we may choose some r such that $0 \leq |\sin(n\pi/7)| < r < 1$ for all n. Then

$$\sum |\sin(n\pi/7)^n| = \sum |\sin(n\pi/7)|^n < \sum r^n < \infty,$$

so the series of absolute values converges, i.e. the series converges absolutely. It is a fact (see corollary 14.7) that absolutely convergent series are convergent.

15.4

(a) The series diverges. Note that $\log n < \sqrt{n}$ for all n. (To see this, check that $f(x) = \sqrt{x} - \log x$ is positive at 1 and always has positive derivative.) Then $\frac{1}{\log n} > \frac{1}{\sqrt{n}}$, so $\frac{1}{\sqrt{n}\log n} > \frac{1}{n}$. By comparison with the harmonic series, the given series diverges.

(b) The series diverges. Since $\frac{\log n}{n} > \frac{1}{n}$, this series also diverges by comparison with the harmonic series. Alternatively, one can evaluate the integral

$$\int_1^N \frac{\log x}{x} dx = \frac{\log^2 x}{2}\Big|_1^N = \frac{\log^2 N}{2},$$

which diverges to $+\infty$ as $N \to \infty$.

(c) The series diverges by the integral test:

$$\int_4^N \frac{1}{x(\log x)(\log\log x)} dx = \log\log\log x\Big|_4^N.$$

$\log\log\log N$ diverges to $+\infty$ as $N \to \infty$.

(d) The series converges. Again using $\log n < \sqrt{n}$, we have $\frac{\log n}{n^2} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$. This is a p-series with $p = 3/2 > 1$, so it converges, and therefore the original series converges by comparison. Alternatively, one can evaluate the integral

$$\int_1^N \frac{\log x}{x^2} dx = -\frac{1}{x} - \frac{\log x}{x}\bigg|_1^N = 1 - \frac{1}{N} - \frac{\log N}{N}.$$

As $N \to \infty$ this converges to 1.

15.6

(c) The series $\sum(-1)^n \frac{1}{\sqrt{n}}$ converges by the alternating series test, but its squared terms form the harmonic series which diverges.