MAT 319 - Spring 2016 Homework 3 Solutions

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Section 8

8.2 (a) $\lim \dfrac{n}{n^2+1} = 0$.

Proof: We know that $\lim 1/n = 0$. Thus: $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N : \left|\dfrac{1}{n}\right| < \varepsilon$.

But since $\dfrac{n}{n^2+1} < \dfrac{n}{n^2} = \dfrac{1}{n}, \forall n \in \mathbb{N}$, it follows that:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N : \left|\frac{n}{n^2+1}\right| = \left|\frac{n}{n^2+1} - 0\right| \leq \left|\frac{1}{n}\right| < \varepsilon$$

Therefore, the limit is indeed 0 by definition.

(b) $\lim \dfrac{7n-19}{3n+7} = 7/3$.

Proof: Given $\varepsilon > 0$, then $\left|\dfrac{7n-19}{3n+7} - \dfrac{7}{3}\right| < \varepsilon$ if and only if $\left|\dfrac{-106}{3(3n+7)}\right| < \varepsilon$. Obviously, $3n+7 > 0$, whence $\left|\dfrac{-106}{3(3n+7)}\right| = \dfrac{106}{3(3n+7)}$. The rest of the proof is quasi-identical to that of the *Discussion* of **Example 2** in page 40 of the textbook: "solve" for n, and then, knowing that the steps are reversible, restate your proof adequately.

(c) $\lim \dfrac{4n+3}{7n-5} = 4/7$.

Proof: Given $\varepsilon > 0$, then $\left|\dfrac{4n+3}{7n-5} - \dfrac{4}{7}\right| < \varepsilon$ if and only if $\left|\dfrac{41}{7(7n-5)}\right| < \varepsilon$. The rest of the proof is quasi-identical to that of the *Discussion* of **Example 2** in page 40 of the textbook (cf. remark above).

(d) $\lim \dfrac{2n+4}{5n+2} = 2/5$.

Proof: Same process as in (b) and (c).

(e) $\lim \dfrac{\sin(n)}{n} = 0$.

We know that $\lim 1/n = 0$. Therefore: $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N : \left|\dfrac{1}{n}\right| < \varepsilon$. But since $|\sin(n)| \leq 1, \forall n \in \mathbb{N}$, it follows that:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N : \left|\frac{\sin(n)}{n}\right| = \left|\frac{\sin(n)}{n} - 0\right| \leq \left|\frac{1}{n}\right| < \varepsilon$$

Therefore, the limit is indeed 0 by definition.
(Note that (a) and (e) are handled similarly using estimates: that is, we compare the given sequence to an already known sequence, and we conclude by comparison. This will also be used in the last problem.)

8.4 Let (t_n) be a bounded sequence and (s_n) be such that $\lim s_n = 0$. As we need to show that $\lim(s_n t_n) = 0$, we formally need to prove that given $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|s_n t_n| < \varepsilon$ for all $n > N$. Now as (t_n) is bounded, $\exists M > 0 : |t_n| \leq M, \forall n \in \mathbb{N}$. Also, since $\lim s_n = 0$, then by definition: $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N : |s_n| < \frac{\varepsilon}{M}$. (Note that the usual ε in the definition can be indeed chosen to be ε/M since ε is arbitrarily small.) Then, we can see that given ε sufficiently small and n large enough, $|s_n t_n| < \varepsilon$ as follows:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N : |s_n t_n| = |s_n||t_n| \leq |s_n|M < \frac{\varepsilon}{M}M = \varepsilon$$

Therefore, $\lim(s_n t_n) = 0$ by definition.

8.6 (a)

$$\lim s_n = 0$$

if and only if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N : |s_n - 0| < \varepsilon$$

if and only if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N : |s_n| < \varepsilon$$

if and only if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N : ||s_n|| < \varepsilon$$

if and only if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N : ||s_n| - 0| < \varepsilon$$

if and only if

$$\lim |s_n| = 0$$

(b) It has already been shown in the textbook that for $s_n = (-1)^n$, (s_n) does not converge and so $\lim s_n$ does not exist. However, note that $|s_n| = |(-1)^n| = 1$ and so $\lim |s_n|$ exists and is equal to 1 as $(|s_n|)$ is just the constant sequence of value 1 identically.

Section 9

9.2 Suppose $\lim x_n = 3$ and $\lim y_n = 7$. Then, by the properties of limits:

$$\lim(x_n + y_n) = \lim x_n + \lim y_n = 3 + 7 = 10$$

(Note that this does *not* require the assumption that $y_n \neq 0, \forall n \in \mathbb{N}$.)
Also, since all the y_n are nonzero, then:

$$\lim \left(\frac{3y_n - x_n}{y_n^2} \right) = \lim \left(3\frac{y_n}{y_n^2} - \frac{x_n}{y_n^2} \right) = 3 \lim \frac{1}{y_n} - \frac{\lim x_n}{\lim y_n^2} = \frac{3}{\lim y_n} - \frac{\lim x_n}{(\lim y_n)^2},$$

and so $\lim \left(\dfrac{3y_n - x_n}{y_n^2} \right) = \dfrac{3}{7} - \dfrac{3}{7^2} = \dfrac{18}{49}$.
(Note that this time the assumption that $y_n \neq 0, \forall n \in \mathbb{N}$ is absolutely necessary for otherwise the expression inside the limit would not be defined.)

9.4 Let $s_1 = 1$ and $s_{n+1} = \sqrt{s_n + 1}$ for $n \geq 1$.
(a) $s_1 = 1$, $s_2 = \sqrt{1+1} = \sqrt{2}$, $s_3 = \sqrt{\sqrt{2}+1}$ and $s_4 = \sqrt{\sqrt{\sqrt{2}+1}+1}$.
(b) Assuming s_n converges, denote its limit by ℓ. Then $\lim s_{n+1} = \lim s_n = \ell$ as if n is large enough, so is $n+1$ or one would simply put $m = n+1$ and note that $m \to \infty$ if and only if $n \to \infty$. Now given this, then since $s_{n+1} = \sqrt{s_n + 1}$, it follows that $s_{n+1}^2 = s_n + 1$ and so, by taking the limit, $\ell^2 = \ell + 1$. This equation can easily be solved using the quadratic formula and we can see that either $\ell = \dfrac{1+\sqrt{5}}{2}$ or $\ell = \dfrac{1-\sqrt{5}}{2}$. However, we can actually prove that $\ell \geq 1$. Indeed, $s_1 = 1 \geq 1$, and assuming that $s_n \geq 1$ for a fixed $n \geq 1$, we obtain that $s_{n+1} = \sqrt{s_n + 1} \geq \sqrt{1+1} > 1$, and thus $s_n \geq 1$ for all $n \in \mathbb{N}$, by induction. By taking the limit, $\ell \geq 1$ since the inequality must also hold for n large enough. Therefore, the second solution is dismissed and $\lim s_n = \ell = \dfrac{1+\sqrt{5}}{2}$.

9.6 Let $x_1 = 1$ and $x_{n+1} = 3x_n^2$ for $n \geq 1$.
(a) Let $a = \lim x_n$. By the same reasoning as in 9.4 (b), then taking the limit in the recurrence relation yields $a = 3a^2$ so that $3a^2 - a = 0$ and thus $a(3a-1) = 0$. Therefore, $a = 0$ or $3a = 1$; i.e., $a = 0$ or $a = \dfrac{1}{3}$.
(b) Notice that $x_2 = 3$ and given the recurrence relation, once would expect x_n to be at least 3 for any $n > 1$ (since $x_1 = 1$). The base case is verified, and for a fixed $n \geq 2$, if we assume that $x_n \geq 3$, then $x_{n+1} = 3x_n^2 \geq 3 \times 3^2 > 3$ which shows (by induction) that $x_n \geq 3$ for any $n > 1$. But then, for n large enough, this should also be true, and thus $a \geq 3$ which contradicts the result achieved in (a). Therefore, a doesn't exist.
(c) The explanation here is that the sequence (x_n) does in fact diverge to $+\infty$, in which case the assumption that the limit exists (and is thus finite) in part (a) is invalid, and confirmed by the reasoning established in part (b). Note that since $x_n \geq 1$ for all $n \geq 1$ from the above, then $x_n^2 \geq x_n$ for all $n \geq 1$ and so $x_{n+1} \geq 3x_n$ for all $n \geq 1$. But then, it can (easily) be seen (and proven inductively) that $x_n \geq 3^{n-1} x_1 = 3^{n-1}$ for any $n \geq 1$. The base case is trivially verified, and for a fixed $n \geq 1$, it follows that $x_{n+1} = 3x_n^2 \geq 3 \times (3^{n-1})^2 = 3^{2n-1} \geq 3^n$, which proves the claim. Clearly, $\lim 3^{n-1} = +\infty$ as $3 > 1$. Therefore, by definition, for any $M > 0$, there exists an $N \in \mathbb{N}$ such that for any $n > N$, $3^{n-1} > M$. Then:

$$\forall M > 0, \exists N \in \mathbb{N}, \forall n > N : x_n > M,$$

since $x_n \geq 3^{n-1}, \forall n \geq 1$, and thus $\lim x_n = +\infty$ by definition.