HOMEWORK 10 SOLUTIONS

1.

(a) Assume $f(a) > y > f(b)$. Let $S = \{x \in [a,b] : f(x) < y\}$. Since $b \in S$, this set is nonempty. Then $x_0 = \inf S$ is a real number and lies in $[a,b]$. For each $n \in \mathbb{N}$, $x_0 + 1/n$ is not a lower bound for S so there exists some s_n such that $x_0 \leq s_n < x_0 + 1/n$. By construction, $\lim s_n = x_0$ and $f(s_n) < y$ for all n. Then continuity of f gives

$$f(x_0) = \lim f(s_n) \leq y.$$

Now define the sequence (t_n) by $t_n = \max\{a, x_0 - 1/n\}$. Then $t_n \in [a,b] \setminus S$, so $f(t_n) \geq y$ for all n. Since $\lim t_n = x_0$, continuity gives $\lim_{n\to\infty} f(t_n) = f(x_0)$. Furthermore, $f(t_n) \geq y$ implies that $\lim f(t_n) \geq y$ by exercise 4 of homework 9. Thus

$$f(x_0) \geq y.$$

It follows that $f(x_0) = y$, as was to be shown.

(b) Assume $f(a) > y > f(b)$. Then $-f(a) < -y < -f(b)$, and $-f$ is continuous. Therefore by the known case of the intermediate value theorem, there exists some $x \in (a,b)$ such that $-f(x) = -y$. But then $f(x) = y$, so we're done.

2. I'll first prove a lemma to be used in parts (b) and (d): If (s_n) is a sequence such that $\lim |s_n| = 0$, then $\lim s_n = 0$. To see this, note that $-|s_n| \leq s_n \leq |s_n|$. From the first inequality (and exercise 4 from homework 9),

$$\lim s_n \geq \lim -|s_n| = -\lim |s_n| = 0.$$

And from the second inequality,

$$\lim s_n \leq \lim |s_n| = 0.$$

This proves that $\lim s_n = 0$.

(a) The limit does not exist. Consider the sequences $x_n = -2\pi n$, $y_n = \pi - 2\pi n$. Then $x_n \cos(x_n) = -2\pi n$, which diverges to $-\infty$. However, $y_n \cos(y_n) = (\pi - 2\pi n)(-1) = 2\pi n - \pi$, which diverges to $+\infty$.

(b) $\lim_{x\to 0} x \cos x = 0$. Suppose that x_n is any sequence converging to 0. Then

$$\lim_{n\to\infty} |x_n \cos x_n| \leq \lim_{n\to\infty} |x_n| = 0.$$

By the above lemma, $\lim_{n\to\infty} x_n \cos x_n = 0$, which proves the claim.

(c) $\lim_{x\to 0^+} \frac{\cos x}{x} = +\infty$. Let (x_n) be a positive sequence that converges to 0. Observe that by continuity, $\lim_{n\to\infty} \cos x_n = \cos 0 = 1$, while $\lim_{n\to\infty} 1/x_n = \lim_{n\to\infty} n = +\infty$. By theorem 9.9, $\lim_{n\to\infty} \frac{\cos x_n}{x_n} = +\infty$.

(d) $\lim_{x\to+\infty} \frac{\cos x}{x} = 0$. Suppose that x_n is any sequence diverging to $+\infty$. Then

$$\lim_{n\to\infty} \left|\frac{\cos x_n}{x_n}\right| \leq \lim_{n\to\infty} \left|\frac{1}{x_n}\right| = 0.$$

By the above lemma, $\lim_{n\to\infty} \frac{\cos x_n}{x_n} = 0$.

3.

(1) Every sequence (x_n) converging to 2 with $x_n < 2$ for all n satisfies $\lim_{n\to\infty} f(x_n) = -\infty$.

(2) For every $M > 0$, there exists a $\delta > 0$ such that $x < 2$ and $|x-2| < \delta$ imply $f(x) < -M$.

(1) $\Rightarrow$ (2) We prove the contrapositive. Suppose that (2) does not hold, so for some $M > 0$, the proposition

$$x < 2 \text{ and } |x-2| < \delta \text{ implies } f(x) < -M$$

fails for every $\delta > 0$. For each $\delta = 1/n$, choose $x_n < 2$ such that $|x_n - 2| < \delta$ but $f(x_n) \geq -M$. Then $x_n \to 2$ and $x_n < 2$, but $\lim_{n\to\infty} f(x_n) \geq -M > -\infty$. Therefore (1) does not hold.

(2) $\Rightarrow$ (1). Let (x_n) be any sequence converging to 2 with $x_n < 2$ for all n. For any $M > 0$, there exists a $\delta > 0$ such that $x <$ and $|x-2| < \delta$ imply $f(x) < -M$. Since $x_n \to 2$, there exists some N such that $|x_n - 2| < \delta$ for all $n > N$. Then $f(x_n) < -M$ for all $n > N$, so we conclude that $\lim_{n\to\infty} f(x_n) = -\infty$. Therefore (1) holds.

20.14 Let (x_n) be any positive sequence that converges to 0. Let $M > 0$ be given. Since $\lim x_n = 0$, there exists some N such that $x_n < 1/M$ for all $n > N$. Then $f(x_n) = 1/x_n > M$ for all $n > N$, so $\lim f(x_n) = +\infty$.

Let (x_n) be any negative sequence that converges to 0. Let $M > 0$ be given. Since $\lim x_n = 0$ there exists some N such that $x_n > -1/M$ for all $n > N$. (This is effectively saying that $|x_n - 0| < 1/M$.) Then $f(x_n) < -M$ for all $n > N$ so $\lim f(x_n) = -\infty$.

20.16

(a) Consider $f_2 - f_1$. Let (x_n) be any sequence converging to a with $x_n > a$ for all n. Then since $(f_2 - f_1)(x_n) \geq 0$, and the limit exists by assumption, problem 4 of the previous assignment shows that $\lim_{n\to\infty}[(f_2 - f_1)(x_n)] \geq 0$. Distributing the limit then shows that $L_2 \geq L_1$.

(b) No. Let $f_1(x) = x$ and $f_2(x) = 2x$. Then $f_1(x) < f_2(x)$ for all $x > 0$, but $\lim_{x\to 0^+} f_1(x) = 0 \not< 0 = \lim_{x\to 0^+} f_2(x)$.

20.18 We first manipulate the expression so that it becomes more manageable.

$$\begin{aligned}
f(x) &= \frac{\sqrt{1+3x^2}-1}{x^2} \\
&= \frac{\sqrt{1+3x^2}-1}{x^2} \cdot \frac{\sqrt{1+3x^2}+1}{\sqrt{1+3x^2}+1} \\
&= \frac{3x^2}{x^2(\sqrt{1+3x^2}+1)} \\
&\simeq \frac{3}{\sqrt{1+3x^2}+1},
\end{aligned}$$

where the $\simeq$ indicates equality when $x \neq 0$. (Note that when $x = 0$, the last expression is well-defined but $f(x)$ is not.) Denote the last expression by $g(x)$, so that $f(x) = g(x)$ for all $x \neq 0$. Now let (x_n) be any sequence that converges to 0 (and $x_n \neq 0$ for all n). Since g is continuous,

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(0) = \frac{3}{2}.$$

The sequence (x_n) is arbitrary, so we conclude that $\lim_{x\to 0} f(x) = 3/2$.

20.20

(a) Let $(x_n) \subseteq S$ be a sequence converging to a. If $\lim_{x \to a^S} f_2(x) = L_2 \neq -\infty$, then $\lim_{n\to\infty} f_2(x_n) = L_2 \neq -\infty$, so

$$\lim_{n\to\infty} (f_1 + f_2)(x_n) = \lim_{n\to\infty} f_1(x_n) + f_2(x_n) = +\infty$$

by exercise 9.11. This proves that $\lim_{x\to a^S}(f_1 + f_2)(x) = +\infty$.

(b) Let $(x_n) \subseteq S$ be a sequence converging to a. If $\lim_{x\to a^S} f_2(x) = L_2 > 0$, then $\lim_{n\to\infty} f_2(x_n) = L_2 > 0$, so

$$\lim_{n\to\infty} (f_1 f_2)(x_n) = \lim_{n\to\infty} f_1(x_n) f_2(x_n) = +\infty$$

by theorem 9.9. This proves that $\lim_{x\to a^S}(f_1 f_2)(x) = +\infty$.

(c) Let $(x_n) \subseteq S$ be a sequence converging to a. If $\lim_{x\to a^S} f_2(x) = L_2 < 0$, then $\lim_{n\to\infty} f_2(x_n) = L_2 < 0$, so we can multiply by -1 to get that $\lim_{n\to\infty} -f_2(x_n) = L_2 > 0$. Now we can again apply theorem 9.9 to get

$$\lim_{n\to\infty} (-f_1 f_2)(x_n) = \lim_{n\to\infty} f_1(x_n)[-f_2(x_n)] = +\infty.$$

Multiplying again by -1, this proves that $\lim_{x\to a^S}(f_1 f_2)(x) = -\infty$.

(d) Nothing. The limit of the quotient might be $+\infty, -\infty$, or even any finite number. To see this, fix $f_2(x) = x$. For the first two cases, take $f_1(x) = \pm 1$ respectively. To obtain the real number r, put $f_1(x) = rx$.

For the sake of completeness, here is a proof of the relevant part from exercise 9.11. Let $(s_n), (t_n)$ be two sequences with $s_n \to +\infty$ and $t_n \to L \neq -\infty$. To see that $s_n + t_n \to \infty$, let $M > 0$ be given. Choose N_1 such that $s_n > M - L$ for all $n > N_1$, and N_2 such that $t_n > L - 1$ for all $n > N_2$. Put $N = \max\{N_1 N_2\}$. Then

$$s_n + t_n > M - 1$$

for all $n > N$, so $s_n + t_n$ can be made arbitrarily large. Thus $s_n + t_n \to +\infty$.