# SERIES

OLGA PLAMENEVSKAYA

ABSTRACT. Class notes for MAT319, Spring 2012. More material is given in the textbook (section 14), but we took a few shortcuts in class to prove (simpler versions of) important theorems.

A series is an infinite sum of the form $a_1 + a_2 + a_3 + a_4 + \dots$, often written as $\sum_{n=1}^{\infty} a_n$ using "sigma-notation". To make sense of the infinite summation, we look at the *partial sums*

$$\begin{aligned} s_1 &= a_1, \\ s_2 &= a_1 + a_2, \\ s_3 &= a_1 + a_2 + a_3, \\ &\dots \\ s_n &= a_1 + a_2 + a_3 + \dots + a_n, \\ &\dots \end{aligned}$$

and consider their sequence $(s_n)$. If the sequence $(s_n)$ converges, we say that the series $\sum a_n$ converges; the limit $\lim s_n = A$ is then called the sum of the series. The sum of the series is often very hard to find (or cannot be expressed as a nice number). In fact, a lot of importnat numbers and functions in mathematics are *defined* to be the sum of a convergent series. Our main goal will be to detect and prove convergence of series.

**Important observation.** Similarly to sequences, convergence/divergence of series depends only on its "tail". Changing finitely many terms in the beginning of the series will not affect convergence (although it may change the sum of the series). Indeed, suppose we replace (some of) the terms $a_1, a_2, \dots a_N$ by different terms $a'_1, a'_2, \dots a'_N$. Let $(s_n)$ be the sequence of partial sums for the series $\sum a_n$, and $(s'_n)$ the partial sums of the new series $a'_1 + a'_2 + \dots a'_N + a_{N+1} + \dots$. Let $S = (a'_1 + a'_2 + \dots a'_N) - (a_1 + a_2 + \dots a_N)$ be the difference between the sum of the new and the old terms (for the terms that got changed). Then whenever we are past all the affected terms, i.e. whenever $n > N$, we have that $s'_n = S + s_n$. By Limit Laws for sequences, it follows that $(s_n)$ and $(s'_n)$ converge or diverge simultaneously; if the old series converges to the sum $A$, the new one will converge to $S + A$.

**Theorem 1.** *If the series $\sum a_n$ converges, then $a_n \to 0$. In other words, if the sequence $(a_n)$ does not converge to* 0 *(i.e. it diverges or converges to a non-zero limit), then the series $\sum a_n$ diverges.*

*Proof.* The two statements in the theorem are equivalent (hello MAT 200). We'll prove the first one. If $\sum a_n$ converges to $A$, the sequence of partial sums $(s_n)$ converges to $A$. Consider the sequence $(s_{n-1})$, where $s_0$ can be defined to be 0, or just left undefined as "only tails matter". Clearly, $(s_{n-1})$ also converges to $A$ (it's essentially the same sequence - why?). Then $s_n - s_{n-1} \to A - A = 0$ by Limit Laws. Since $s_n - s_{n-1} = a_n$, we are done. □

If $\sum a_n$ is a series with *non-negative terms*, i.e. $a_n \geq 0$, more is true: if the terms $a_n$ do not converge to 0, then $\sum a_n = +\infty$. (This is a question in Homework 6).

By contrast, the converse is manifestly not true: if $a_n \to 0$, it **does not** follow that the series will converge. There are plenty of divergent series with terms $a_n$ converging to 0 (harmonic series and $p$-series for $p < 1$ should be familiar from calculus).

Now we prove two tests that guarantee convergence of series, comparison test and ratio test.

**For the remainder of the notes, we consider series with non-negative terms only, i.e. we assume $a_n \geq 0$ for all $n$.** Series $\sum a_n$ with non-negative terms is easier to study, for the following reason. The partial sums $s_n$ of such series form an increasing (or at least non-decreasing),

$$s_1 \leq s_2 \leq s_3 \dots.$$

By the monotone convergence theorem, the sequence $(s_n)$ is guaranteed to converge once we know that it is bounded. Thus, series with non-negative terms converge whenever their partial sums are bounded. Additionally, we know that an increasing bounded sequence converges to its supremum; thus in case of convergence, we have $\sum a_n = \sup s_n$.

**Theorem 2** (Comparison test)**.** *Suppose that $\sum a_n$, $\sum b_n$ are two series with non-negative terms, and $a_n \leq b_n$ for all $n$. Then if the series $\sum b_n$ converges, the series $\sum a_n$ also converges. In this case, if $\sum a_n = A$, $\sum b_n = B$, then $A \leq B$.*

*Proof.* Let $t_n$ stand for partial sums of $\sum b_n$. Since $\sum b_n$ converges, the sequence $(t_n)$ also converges. Therefore, $(t_n)$ is bounded: there is an upper bound $M$ s.t. $t_n \leq M$ for all $n$. Now, notice that because $a_k \leq b_k$ for all $k$, summing up we have that $s_n \leq t_n$ for partial sums. But then $s_n \leq t_n \leq M$, the sequence $(s_n)$ is bounded, and therefore (because it's increasing!) converges.

For the second statement, we have $\sum b_n = \sup t_n = B$. Being its supremum, $B$ is an upper bound for $t_n$; then the previous reasoning shows that $s_n \leq t_n \leq B$, and so $A = \sup s_n \leq B$. □

Note that non-negativity of terms is very important here, comparison test doesn't work otherwise.

To use the comparison test, it's helpful to have a few "favorite" convergent series. In particular, we often use the geometric series $\sum Cq^n$, which converges whenever $0 < q < 1$. (See example 1 on p. 91 in the book) Another useful series is the "telescoping" series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. The name is due to fact that the partial sum $s_n = \frac{1}{1\cdot 2} + \dots \frac{1}{n(n+1)}$ collapses "like a telescope", and we get that $s_n = 1 - \frac{1}{n+1}$ if we write $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ and cancel terms in the resulting expansion. Clearly, then $s_n \to 1$, so the series converges to 1.

Now we will use comparison to prove (a special case of) the ratio test.

**Theorem 3.** *(Ratio Test) Suppose $\sum a_n$ is a series with non-negative terms, such that $\lim \frac{a_{n+1}}{a_n} = L$, and $L < 1$. Then the series converges.*

*Proof.* First, find a small neighborhood of $L$ which is entirely to the left of 1, that is, take $\epsilon > 0$ such that $L + \epsilon < 1$. (This is possible because $L < 1$.) Set $q = L + \epsilon$, then $q < 1$. We will establish convergence by comparing the tail of our series to a geometric series. Indeed, since $\lim \frac{a_{n+1}}{a_n} = L$, there is (why?) some $N$-tail of the sequence where

$$\frac{a_{n+1}}{a_n} < L + \epsilon = q \text{ whenever } n > N.$$

Then, $a_{N+2} < a_{N+1}q$, $a_{N+3} < a_{N+2}q < a_{N+1}q^2$, $a_{N+4} < a_{N+3}q < a_{N+1}q^3$, and so on. Then the series $a_{N+1} + a_{N+2} + a_{N+3} + \dots$ converges by comparison with the geometric series $a_{N+1}(1 + q + q^2 + q^3 + \dots)$. Since we know that finitely many terms $a_1, \dots a_N$ do not affect convergence, the series $\sum a_n$ also converges.

□

More results (in particular,tests for divergence) are given in Homework 6.