MAT 319, Spring 2012
Solutions to HW 6

1. Let (x_n) be a sequence of nonnegative terms, $x_n \geq 0$ for all n. Suppose that (x_n) does not converge to 0. Prove that one can always find a subsequence (x_{n_k}) such that $x_{n_k} > \epsilon$ for all terms in the subsequence (ϵ may not be arbitrary here).
   Proof: We must carefully negate the statement "$(x_n) \to 0$" to find an ϵ that we can work with. "$(x_n) \to 0$" means that every neighborhood of 0 contains some tail of (x_n). Therefore, if we say that (x_n) does NOT converge to 0, we are saying that there exists some ϵ-neighborhood of 0 that does not contain ANY tail of (x_n).
   Let us pick an $\epsilon > 0$ satisfying this. Since no tail of (x_n) is contained in this ϵ-nbhd, we can surely find at least 1 term, let's call it x_{n_1}, outside this neighborhood: $|x_{n_1}| \geq \epsilon$. Now that we have a first term, we can proceed by induction. Let's do one more to see how the inductive step will go: To define x_{n_2}, let's look at the n_1-tail of (x_n). Recall that by definition, this tail starts with the term AFTER x_{n_1}. This tail isn't contained in the ϵ-nbhd either, so we can find a term x_{n_2} from this tail satisfying $|x_{n_2}| \geq \epsilon$. Since x_{n_2} came from the n_1 tail, we know that $n_2 > n_1$.
   Inductive step: Suppose that x_{n_k} is defined for some k. We need to define $x_{n_{k+1}}$. The n_k-tail of (x_n) isn't contained in our ϵ-nbhd, so we can find a term, call it $x_{n_{k+1}}$, from this tail satisfying $\left|x_{n_{k+1}}\right| \geq \epsilon$. It also follows that $n_k < n_{k+1}$.
   Now we have inductively defined our subsequence $(x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots)$, and it is a well-defined subsequence since $n_1 < n_2 < \cdots < n_k < \cdots$. Furthermore, since all the terms are nonnegative, we know that $x_{n_k} = |x_{n_k}| \geq \epsilon$. The conclusion now follows by keeping this same subsequence and shrinking ϵ slightly to a smaller positive number, thereby making the inequality strict.

2. Suppose that a sequence (x_n) is not bounded below. Prove that it has a subsequence diverging to $-\infty$. Give an example to show that the entire sequence may not diverge to $-\infty$.
   Proof: We will construct such a subsequence term by term. The idea is the following: We know that the sequence $(-k)$ diverges to $-\infty$. Maybe we can find a subsequence such that $x_{n_k} < -k$ for all k. By an exercise from a previous homework (3b in HW 4), this would imply the result. It seems like it should be possible: since none of the numbers $-k$ is a lower bound for the sequence, we can find some term smaller than it. But we should be careful in writing it up.
   Since -1 is not a lower bound, we can find a term, call it x_{n_1} satisfying $x_{n_1} < -1$. x_{n_2} is trickier to find because it must come from the n_1-tail. We could try choosing it so that $x_{n_2} < -2$, but this doesn't guarantee that $n_1 < n_2$. In fact, even choosing $x_{n_2} < \min\{-2, x_{n_1}\}$ is not enough. We need to choose
$$x_{n_2} < \min\{-2, x_1, x_2, \ldots, x_{n_1}\}.$$
   In this way, we can be assured that $x_{n_2} < -2$, and also it is smaller than x_{n_1} and all preceding terms. Therefore, it must come from the n_1-tail as desired.
   Inductive step. Suppose now that x_{n_k} is defined. Let $m_{k+1} = \min\{-(k+1), x_1, x_2, \ldots, x_{n_k}\}$. m_{k+1} cannot be a lower bound for (x_n), so we can find a term $x_{n_{k+1}} < m_{k+1}$. Since this implies that $x_{n_{k+1}}$ is less than all term preceding x_{n_k}, it must come from the n_k-tail: $n_k < n_{k+1}$. Also, we have $x_{n_{k+1}} < -(k+1)$.
   This recursive definition gives a well-defined subsequence (x_{n_k}) satisfying $x_{n_k} < -k$ for all positive integers k. Exercise 3b in HW 4 completes the proof.

   As an example of a sequence that is not bounded below, and also does not diverge to $-\infty$, look at
$$x_n = \begin{cases} -n & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$
   The odd terms diverge to $-\infty$, so this sequence is definitely not bounded below. However, the even terms are all 0, so the entire sequence cannot diverge to $-\infty$.

3. Suppose that $\sum a_n$ is a series with non-negative terms, $a_n \geq 0$ for all n. Suppose that $\sum a_n$ does not converge. Show that $\sum a_n$ diverges to $+\infty$, that is, for any α there is an N such that $s_n > \alpha$ for all

partial sums s_n with $n > N$.
Proof: We must show that the sequence of partial sums (s_n) diverges to $+\infty$. Notice first that this sequence is non-decreasing. Indeed, for each n, $s_{n+1} - s_n = a_{n+1} \geq 0$, proving that $s_n \leq s_{n+1}$ for all n. If the sequence (s_n) were bounded, then in must be convergent, because it is nondecreasing. However, we know the sequence of partial sums diverges–this is what it means for a series to diverge. Therefore, (s_n) is unbounded and nondecreasing. Since s_1 is a lower bound, (s_n) must be unbounded above.
Let $\alpha > 0$. Since α cannot be an upper bound, we can find some partial sum satisfying $s_N > \alpha$. Since the sequence of partial sums is nondecreasing, the entire N-tail of partial sums must lie above α.

$$\alpha < s_N \leq s_{N+1} \leq s_{N+2} \leq s_{N+3} \leq \cdots .$$

4. Suppose that $\sum a_n$, $\sum b_n$ are two series with nonnegative terms, $\sum b_n$ diverges, and $a_n \geq b_n$. Prove that $\sum a_n$ diverges.
Proof: Let (s_n) be the sequence of partial sums for $\sum a_n$, and (t_n) the same for $\sum b_n$. Since $\sum b_n$ is a divergent series with nonnegative terms, we know that (t_n) diverges to $+\infty$ from the previous problem. For any given n, we have:

$$s_n = \sum_{k=1}^{n} a_k \geq \sum_{k=1}^{n} b_k = t_n.$$

Therefore, by problem 3a on Homework 4, (s_n) must diverge to $+\infty$ as well. Therefore, $\sum a_n$ diverges.

5. Question 14.14:
Let us first examine the sequence $(a_n)_{n=2}^{\infty}$ given in the book:

$$(a_n)_{n=2}^{\infty} = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \ldots\right).$$

This sequence is defined as follows: for any $n \geq 2$, there is some integer k such that

$$2^{k-1} < n \leq 2^k,$$

a_n is then defined to be 2^{-k}. (By the way, this automatically shows that $\frac{1}{n} \geq a_n$, a fact that will be useful later.) To examine the behavior of the series $\sum a_n$, we look at the sequence of partial sums (s_n). Since all the terms a_n are positive, the sequence (s_n) must be increasing. Let's look at some particular partial sums:

$$\begin{aligned}
s_2 &= \sum_{n=2}^{2} a_n = \frac{1}{2} \\
s_4 &= \sum_{n=2}^{4} a_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = \frac{2}{2} \\
s_8 &= \sum_{n=2}^{8} a_n = \frac{1}{2} + \frac{2}{4} + \frac{4}{8} = \frac{3}{2}.
\end{aligned}$$

In general, there are 2^{n-1} terms equal to $\frac{1}{2^n}$; adding these up will contribute $\frac{1}{2}$. Therefore, $s_{2^n} = \frac{n}{2}$. Now it is easy to prove that (s_n) diverges to $+\infty$. Let $\alpha > 0$. We must find a tail of (s_n) contained entirely in $(\alpha, +\infty)$. To do this, choose an integer $N > 2\alpha$ so that $s_{2^N} = \frac{N}{2} > \alpha$. Now that we have 1 term greater than α, the fact that (s_n) is an increasing sequence proves that an entire tail (the 2^N-tail, to be precise) is greater than α:

$$\alpha < s_{2^N} < s_{2^N+1} < s_{2^N+2} < \cdots .$$

Thus, (s_n) diverges to $+\infty$ as desired.

So far we have shown that $\sum a_n$ diverges to $+\infty$. Also, we stated before that $\frac{1}{n} \geq a_n$. Therefore, by problem 3 above, $\sum \frac{1}{n}$ diverges to $+\infty$ as well.

6. Suppose that $\lim \frac{a_{n+1}}{a_n} = L$, and $L > 1$. Prove that the series $\sum a_n$ diverges, assuming $a_n > 0$ for all n.

Since $1 < L$, we claim that some tail of the sequence $\left(\frac{a_{n+1}}{a_n}\right)$ is always greater than 1. To see this, let $\epsilon = L - 1 > 0$. We can find an integer N such that if $n \geq N$, then $L - \epsilon < \frac{a_{n+1}}{a_n} < L + \epsilon$; in particular, $1 < \frac{a_{n+1}}{a_n}$. Therefore,

$$\begin{aligned} a_{N+1} &> a_N \\ a_{N+2} &> a_{N+1} > a_N \\ a_{N+3} &> a_{N+2} > a_{N+1} > a_N \end{aligned}$$

and so on; i.e. $a_{N+k} > a_N$. As the N-tail is always greater than a_N, a positive number, the sequence of terms (a_n) cannot possibly converge to 0. Therefore, the divergence test implies $\sum a_n$ diverges.