MAT 319, Spring 2012
Solutions to HW 5

1. It was shown in class that $2^n > n$ for all positive integers n.

   (a) Since $2^n > n$, and n diverges to $+\infty$, Problem 3a on Homework 4 implies that (2^n) diverges to $+\infty$ as well.
   For a more basic proof coming straight from the definitions: Let $\alpha > 0$. We must find a tail of (2^n) that is always greater than α. Let $N > \alpha$. If $n > N$, then

   $$2^n > n > N > \alpha.$$

   So the N-tail satisfies our requirement.

   (b) Prove that $\left(\frac{1}{2^n}\right)$ converges to 0.
   Since (2^n) is a sequence of positive terms diverging to $+\infty$, $\left(\frac{1}{2^n}\right)$ converges to 0.

2. Prove that $\sqrt{5}$ is irrational.
Suppose that $\sqrt{5}$ is rational. Then we can express $\sqrt{5}$ uniquely as $\frac{p}{q}$, where p, q are coprime positive integers. Therefore, $p^2 = 5q^2$, so p^2 is a multiple of 5. Since 5 is prime, this means that p must also be a multiple of 5: $p = 5k$ for some positive integer k. Thus,

$$5q^2 = p^2 = (5k)^2 = 25k^2.$$

Therefore, $q^2 = 5k^2$, showing that q^2, whence q, is also a multiple of 5. Thus, we have shown that p and q are both multiples of 5, contradicting the fact that they are coprime.

3. Let (x_n) be an increasing sequence.

   (a) Prove that (x_n) is bounded below.
   Since (x_n) is increasing, x_1 is a lower bound. Therefore (x_n) is bounded below.
   For a very rigorous proof of this fact, we can proceed by induction: We know that $x_n < x_{n+1}$ for each positive integer n. Thus, $x_1 < x_2$. Suppose $x_1 < x_k$ for some integer k. Then $x_1 < x_k < x_{k+1}$, and so $x_1 < x_{k+1}$ as well. By induction, $x_1 \leq x_n$ for all n. QED

   (b) Suppose that (x_n) is not bounded above. Prove that (x_n) diverges to $+\infty$.
   Let $\alpha > 0$. We must find a tail of (x_n) that is greater than α. We start by finding a single element greater than α. Since (x_n) is not bounded above, α cannot be an upper bound for the sequence. Therefore, there must exist some x_N that is greater than α. To conclude the argument, we remember that (x_n) is increasing. So

   $$\alpha < x_N < x_{N+1} < x_{N+2} < \cdots.$$

   In particular, the N-tail of (x_n) is always greater than α.

4.

   (a) Prove that $1 \neq 0$.
   In this proof, the assumption that the set of real numbers has more than 1 element is 100% necessary.
   Assume, for contradiction, that $1 = 0$. Let $a \in \mathbb{R}$. By Axiom M3 and Theorem 3.1(ii),

   $$a = a \cdot 1 = a \cdot 0 = 0.$$

   Thus, we have shown that 0 is the only element of $\mathbb{R}$, contradicting the assumption that R has at least two elements.

(b) Prove that $0 \leq 1$.
Assume that $1 \leq 0$. We apply Axiom O4 to add -1 to both sides of the inequality to get $0 \leq -1$. Since -1 is therefore "non-negative", we can apply Axiom O5 to multiply $1 \leq 0$ by -1:

$$\begin{aligned} 1(-1) &\leq 0(-1) \\ -1 &\leq 0. \end{aligned}$$

Hence, $0 \leq -1$ and $-1 \leq 0$. Therefore, by Axiom O2, $0 = -1$. Adding 1 to this equality yields $1 = 0$. Thus we have proven that either $0 = 1$ or our assumption was flawed and in fact $0 < 1$. In either case, $0 \leq 1$.

5. **Question 4.6** Let S be a nonempty bounded subset of $\mathbb{R}$.

(a) Prove that $\inf S \leq \sup S$.
Let $s \in S$. Since $\inf S$ is a lower bound of S, and $\sup S$ is an upper bound,

$$\inf S \leq s \leq \sup S.$$

(b) What can you say about S if $\inf S = \sup S$?
We showed above that $\inf S \leq s \leq \sup S$ for all elements $s \in S$. Let $A := \inf S = \sup S$, so that $A \leq s \leq A$ for all $s \in S$. This can only be true if $A = s$, and so S has only one element, namely A.

6. **Question 4.7** Let S and T be nonempty bounded sets of $\mathbb{R}$.

(a) Prove that if $S \subseteq T$, then $\inf T \leq \inf S \leq \sup S \leq \sup T$.
$\sup T$ is an upper bound for T, and $\inf T$ is a lower bound. Thus, for every $t \in T$, $\inf T \leq t \leq \sup T$. If $s \in S$, then $s \in T$ as well, and so $\inf T \leq s \leq \sup T$. Therefore, $\sup T$ is an upper bound for S, and $\inf T$ is a lower bound. Since $\sup S$ is the LEAST upper bound for S, $\sup S \leq \sup T$. Likewise, $\inf T \leq \inf S$ because $\inf S$ is the GREATEST lower bound of S. Combining this with part (a) of the last question shows

$$\inf T \leq \inf S \leq \sup S \leq \sup T$$

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