MAT 319 Introduction to Analysis

Homework 4

due Thursday, February 23

Please prove (or explain as appropriate) all your answers.

Question 1. Prove the following:
(a) If the sequence (x_n) is bounded below, and (y_n) diverges to $+\infty$, then (x_n+y_n) diverges to $+\infty$. (Note that (x_n) may or may not converge.)

(b) If the sequence (x_n) diverges to $+\infty$, it is bounded below.

(c) If both sequences (x_n) and (y_n) diverge to $+\infty$, then $(x_n + y_n)$ diverges to $+\infty$.

Hints and Comments: (a) is quite similar to what we did in class. The proof of (b) follows the lines of the proof that convergent sequences are bounded. The part (c) can be derived from (a) and (b).

Question 2. (a) Let (x_n) and (z_n) be two sequences such that $z_n = -x_n$ for every n. Prove that x_n diverges to $+\infty$ **if and only if** z_n diverges to $-\infty$. **MAT 200 reminder:** "if and only if" means you have to check two directions.

(b) Suppose that x_n diverges to $-\infty$, and x_n is bounded above. Show that $(x_n + y_n)$ diverges to $-\infty$.

(c) Suppose that both sequences (x_n) and (y_n) diverge to $-\infty$. Prove that $(x_n + y_n)$ diverges to $-\infty$.

Hints and Comments: You will need to prove (a) from definitions. Then, try to use the previous problem (together with part (a)) for a quick proof for (b) and (c).

Question 3. (a) Suppose that x_n diverges to $+\infty$, and $y_n \geq x_n$ for every n. Prove that (y_n) also diverges to $+\infty$.

(b) Suppose that x_n diverges to $-\infty$, and $y_n \leq x_n$ for every n. Prove that (y_n) also diverges to $-\infty$.

Question 4. (a) Suppose that (x_n) converges to 0, and (y_n) is bounded. Prove that $(x_n y_n)$ converges to 0.

(b) If (x_n) converges to 0 and (y_n) diverges to $+\infty$, the behavior of the product sequence $(x_n y_n)$ is unpredictable. Give examples (with justifications when needed) showing that $(x_n y_n)$ may converge to a finite limit, diverge to an infinite limit, or have no (finite or infinite) limit at all.

Question 5. Find the following limits. Use any method, but please justify every step.

(a)

$$x_n = \sqrt[3]{n^2 - n - 1}$$

(b)

$$x_n = \frac{7n^2 + n^2 + 1}{\sqrt{n^4 - 5n^3 - 3}}$$