MAT 319, Spring 2012
Solutions to HW 4

1. Prove the following:

   (a) If the sequence (x_n) is bounded below, and (y_n) diverges to $+\infty$, then $(x_n + y_n)$ diverges to $+\infty$.
   Proof: Let α be any number. We must find a tail of $(x_n + y_n)$ that is always greater than α. Let B be a lower bound for (x_n) so that $x_n \geq B$ for all n. There exists a tail of (y_n) that is always greater than $\alpha - B$. That is: $\exists N$ such that if $n > N$, then $y_n > \alpha - B$. Therefore, if $n > N$, then
   $$x_n + y_n > B + (\alpha - B) = \alpha.$$
   Hence, $(x_n + y_n)$ diverges to $+\infty$ as desired.

   (b) If the sequence (x_n) diverges to $+\infty$, then it is bounded below.
   Proof: Since (x_n) diverges to infinity, we can always find a tail of the sequence whose terms are as large as we want. In particular, $\exists N$ such that $x_n > 1000$ whenever $n > N$. (The 1000 is completely arbitrary. I could have picked any other number and found a corresponding tail. Picking one number is just a way to give us something definite to work with.) Therefore, 1000 is a lower bound for the N-tail.
   $$1000 < x_{N+1}, x_{N+2}, x_{N+3}, \ldots.$$
   It is not necessarily a lower bound for the entire sequence. To achieve this, let $m = \min\{x_1, x_2, \ldots, x_N, 1000\}$. Then m is obviously less than or equal to $x_1, x_2, \ldots, x_N$, and since it is also no bigger than 1000, it must be less than the terms in the N-tail as well. Therefore, m is a lower bound for the entire sequence.

   (c) If both sequences (x_n) and (y_n) diverge to $+\infty$, then $(x_n + y_n)$ diverges to $+\infty$.
   Since (x_n) diverges to $+\infty$, it is bounded below by 1b. 1a then implies that $(x_n + y_n)$ diverges to $+\infty$.

2.

   (a) Let (x_n) and (z_n) be two sequences such that $z_n = -x_n$ for every n. Prove that x_n diverges to $+\infty$ if and only if z_n diverges to $-\infty$.
   $\Rightarrow$) Assume $(x_n) \to +\infty$. Let $\alpha < 0$. We want to find a tail of (z_n) that is always SMALLER than α, a tail in the "neighborhood" $(-\infty, \alpha)$ of $-\infty$. We know that we can find a tail of (x_n) that is bigger than any given value, so in particular, we can find an N-tail such that if $n > N$, then $x_n > -\alpha$. Now we can just multiply this inequality by -1. Therefore, if $n > N$, then $-x_n < \alpha$; i.e. $z_n < \alpha$. Thus, the N-tail of (z_n) is contained in the set $(-\infty, \alpha)$. Therefore, $(z_n) \to -\infty$.
   $\Leftarrow$) Assume $(z_n) \to -\infty$. Let $\alpha > 0$. Now we want to find a tail of (x_n) that is larger than α. We know that we can find a tail of (z_n) as small as we like, so we use the "negative one" trick again: We can find an N-tail such that if $n > N$, then $z_n < -\alpha$. Therefore, if $n > N$, then $-z_n > \alpha$; i.e. $x_n > \alpha$. Hence, the N-tail of (x_n) is contained in the "neighborhood of $+\infty$" $(\alpha, +\infty)$. Therefore, $(x_n) \to +\infty$.

   This result is very useful. If we have a theorem saying that sequences satisfying certain conditions tend to $+\infty$, this result gives us an analogous theorem showing certain other sequences converge to $-\infty$ for free. All we have to do is multply everything by -1 to obtain this new theorem. We will see how this works in the other

   (b) Suppose (x_n) diverges to $-\infty$, and (y_n) is bounded above. Show that $(x_n + y_n)$ diverges to $-\infty$.
   Proof: This result looks a lot like that in 1a, except with terms going to $-\infty$ instead of $+\infty$. We could go through the entire proof of 1a again, or we could multiply everything by -1 and apply 2a and 1a. This is the tack we take.
   First, we note that $(-y_n)$ is bounded below: if b is an upper bound for (y_n), then $-b$ is a lower bound for $(-y_n)$. Also, by 2a, $(-x_n)$ diverges to $+\infty$. Therefore, 1a implies that $(-x_n - y_n)$ diverges to $+\infty$. Multiplying by -1 again and applying 2a yields the desired result.

(c) Suppose (x_n) and (y_n) both diverge to $-\infty$. Prove that $(x_n + y_n)$ diverges to $-\infty$.
Proof: 2a implies that both $(-x_n)$ and $(-y_n)$ diverge to $+\infty$. Therefore, 1c implies that $(-x_n - y_n)$ diverges to $+\infty$. Applying 2a again yields the result that $(x_n + y_n)$ diverges to $-\infty$.

3.

(a) Suppose that x_n diverges to $+\infty$, and that $y_n \geq x_n$ for every n. Prove that y_n also diverges to $+\infty$.
Proof: Let $\alpha > 0$. We want a tail of (y_n) contained in $(\alpha, +\infty)$. We know we can find a tail of (x_n) contained in this set. There exists a number N such that whenever $n > N$, $x_n > \alpha$. Therefore, for these same values of n, $y_n \geq x_n > \alpha$. Since α was arbitrary, y_n also diverges to $+\infty$.

(b) Suppose that x_n diverges to $-\infty$, and that $y_n \leq x_n$ for every n. Prove that y_n also diverges to $-\infty$.
Proof: We could repeat the proof of 3a, reversing all the correct inequalities, or we could use 2a again. By 2a, $(-x_n)$ diverges to $+\infty$. Also, $-y_n \geq -x_n$ for every n. Hence, by 3a, $(-y_n)$ also diverges to $+\infty$. Then 2a implies that (y_n) diverges to $-\infty$.

4.

(a) Suppose that (x_n) converges to 0 and (y_n) is bounded. Prove that $(x_n y_n)$ converges to 0.
Proof: Let $\epsilon > 0$. Let $B > 0$ be a bound for (y_n), so that $|y_n| \leq B$ for all n. We can find a number N such that if $n > N$, then $|x_n - 0| = |x_n| < \frac{\epsilon}{B}$. Thus, if $n > N$, then

$$|x_n y_n - 0| = |x_n y_n| = |x_n|\,|y_n| < \frac{\epsilon}{B} B = \epsilon.$$

Since ϵ was arbitrary, this shows $(x_n y_n) \to 0$.

(b) If $x_n = \frac{1}{n}$ and $y_n = n$, then $(x_n) \to 0$, $(y_n) \to +\infty$, and $(x_n y_n)$ converges to the finite limit 1. (In fact, it is the constant sequence.)
If $x_n = \frac{1}{n}$ and $y_n = n^2$, then $(x_n) \to 0$, $(y_n) \to +\infty$, but $x_n y_n = n$, so that $(x_n y_n)$ diverges to $+\infty$.
If $x_n = (-1)^n / n$, then $(x_n) \to 0$. (For a proof of this, one can appeal to 4a, as x_n is the product of a bounded sequence, $(-1)^n$, and a sequence converging to 0, $1/n$.) Let $y_n = n$, so that $(y_n) \to +\infty$. Then $(x_n y_n) = ((-1)^n)$ is a sequence that has no finite or infinite limit.
If you prefer to only work with positive sequences, set $x_n = \frac{1}{n}$, and set

$$y_n = \begin{cases} n & \text{if } n \text{ is odd} \\ n^2 & \text{if } n \text{ is even.} \end{cases}$$

Then y_n still tends toward $+\infty$, but

$$x_n y_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}$$

does not converge or tend to $+\infty$.

5. Find the following limits. Justify each step.

(a) $x_n = \sqrt[3]{n^2 - n - 1}$.
This sequence diverges to $+\infty$. One method of justifying this is by comparison: If $n \geq 3$, then $n^2 - n - 1 \geq n$. In fact, the quadratic formula tells us that $n^2 - 2n - 1 \geq 0$ whenever$n \geq 1 + \sqrt{2}$. Thus, if we restrict our attention to $n \geq 3 > 1 + \sqrt{2}$, then $x_n \geq n^{1/3}$. Since $n^{1/3}$ diverges to $+\infty$, 3a implies x_n tends to $+\infty$ as well.

(b) $x_n = \dfrac{7n^2 + n + 1}{\sqrt{n^4 - 5n^3 - 3}}$.
We rewrite x_n in the equivalent form

$$x_n = \frac{7 + \frac{1}{n} + \frac{1}{n^2}}{\sqrt{1 - \frac{5}{n} - \frac{3}{n^4}}}.$$

We know that $\left(\frac{1}{n^p}\right) \to 0$ for all $p > 0$, and furthermore, we can multiply by any constant c so that $\left(\frac{c}{n^p}\right) \to 0$. Therefore, $\left(\frac{1}{n}\right), \left(\frac{1}{n^2}\right), \left(-\frac{5}{n}\right), \left(-\frac{3}{n^4}\right)$ all converge to 0. We now apply the limit sum formula to see $\left(7 + \frac{1}{n} + \frac{1}{n^2}\right) \to 7$ and $\left(1 - \frac{5}{n} - \frac{3}{n^4}\right) \to 1$. Since $1 - \frac{5}{n} - \frac{3}{n^4}$ converges to a positive number, it must be eventually positive, (in fact it is positive for $n \geq 6$), and so we can apply our root rule to see $\left(\sqrt{1 - \frac{5}{n} - \frac{3}{n^4}}\right) \to \sqrt{1} = 1$. Finally, we can apply the limit quotient rule because the numerator and denominator both converge – and the denominator does not converge to 0:

$$\lim (x_n) = \frac{7}{1} = 7.$$