MAT 319, Spring 2012
Solutions to HW 11

1. Suppose f is differentiable at x_0, and $f'(x_0) < 0$. Prove that in some neighborhood of x_0, we have

$$\begin{aligned} f(x) &< f(x_0) \text{ for every } x < x_0, \\ f(x) &> f(x_0) \text{ for every } x > x_0. \end{aligned}$$

$$0 > f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Therefore, there is some δ such that when $0 < |x - x_0| < \delta$, $\frac{f(x)-f(x_0)}{x-x_0} < 0$. Therefore, if $x_0 - \delta < x < x_0$, then $x - x_0 < 0$ and $\frac{f(x)-f(x_0)}{x-x_0} < 0$, so $f(x) - f(x_0) > 0$; $f(x) > f(x_0)$. Similarly, if $x_0 < x < x_0 + \delta$, then $x - x_0 > 0$, and so $f(x) - f(x_0) < 0$; $f(x) < f(x_0)$.

2. (29.7)

   (a) Suppose that f is twice differentiable on an open interval I and that $f''(x) = 0$ for all $x \in I$. Show that f has the form $ax + b$ for suitable constants a and b.
   Since $(f')'(x) = 0$, we know that $f'(x)$ is a constant function, from Corollary 29.4. Thus, $f'(x) = a$. Hence,

   $$(f(x) - ax)' = f'(x) - a = 0.$$

   Applying Corollary 29.4 again shows that there exists a constant b such that $f(x) - ax = b$. Thus, $f(x) = ax + b$ for some constants a and b as desired.

   (b) Suppose f is three times differentiable on an open interval I and that $f''' = 0$ on I. What form does f have? Prove your answer.
   $f(x) = ax^2 + bx + c$ for some constants a, b, c. To prove this, note that $(f')'' = 0$, so we can apply part (a) to f': There are constants a and b such that $f'(x) = ax + b$. Thus,

   $$\left(f(x) - \frac{a}{2}x^2 - bx\right)' = f'(x) - ax - b = 0.$$

   So by Corollary 29.4, there exists a constant c such that $f(x) = \frac{a}{2}x^2 + bx + c$. Since $\frac{a}{2}$ is just another constant, we can rename it a, and the claim is proved.

3. (29.13) Prove that if f and g are differentiable on $\mathbb{R}$, if $f(0) = g(0)$, and if $f'(x) \le g'(x)$ for all $x \in \mathbb{R}$, then $f(x) \le g(x)$ for $x \ge 0$.
Let $h(x) = g(x) - f(x)$. Then h is differentiable, and $h'(x) = g'(x) - f'(x) \ge 0$. Thus, by Corollary 29.7(c), h is an increasing function. If $x \ge 0$, then $h(x) \ge h(0) = g(0) - f(0) = 0$. Therefore, $g(x) - f(x) \ge 0$, and so $g(x) \ge f(x)$.

4. (29.14) Suppose that f is differentiable on $\mathbb{R}$, that $1 \le f'(x) \le 2$ for $x \in \mathbb{R}$, and that $f(0) = 0$. Prove that $x \le f(x) \le 2x$ for all $x \ge 0$.
This follows immediately from the last problem.
Taking the functions $f(x)$ and $2x$ to play the roles of f and g in the last problem, respectively, we see that $f(0) = 0 = 2(0)$, and $f'(x) \le 2 = (2x)'$. Therefore, $f(x) \le 2x$ for all $x \ge 0$.
Similarly, if x and $f(x)$ play the roles of f and g, respectively, we can conclude that $x \le f(x)$.

5. (29.5) Let f be defined on $\mathbb{R}$, and suppose that $|f(x) - f(y)| \le (x - y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is a constant function.
We will show that f' vanishes identically. Let $a \in \mathbb{R}$ be arbitrary. For $x \ne a$, we have

$$0 \le \left| \frac{f(x) - f(a)}{x - a} \right| \le |x - a|.$$

Therefore, if x is in an ϵ-neighborhood of a, $\frac{f(x)-f(a)}{x-a}$ is in the ϵ-neighborhood of 0. Thus,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists, and is equal to 0. Since $f'(a)$ is zero everywhere, Corollary 29.4 tells us that f is constant.