MAT 319, Spring 2012
Solutions to HW 10

1.

(a) Prove that a constant function is differentiable at any point. Find its derivative.
Let $f(x) = c$ for all x. Let $a \in \mathbb{R}$. For $x \neq a$,

$$\frac{f(x) - f(a)}{x - a} = \frac{c - c}{x - a} = 0.$$

Therefore, $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists, and is equal to 0.

(b) Suppose that f is differentiable at a. Arguing from definitions, prove that the function $g(x) = 3f(x) + 2$ is differentiable at a.
For $x \neq a$,

$$\frac{g(x) - g(a)}{x - a} = 3\frac{f(x) - f(a)}{x - a}.$$

Since $f'(a)$ exists, and so $\lim_{x\to a} \dfrac{g(x) - g(a)}{x - a} = 3\lim_{x\to a} \dfrac{f(x) - f(a)}{x - a} = 3f'(a)$. g is differentiable at a.

2.

(a) Suppose that $g(x)$ is differentiable at a, and $g(a) \neq 0$. Arguing from definitions, prove that the function $h(x) = \frac{1}{g(x)}$ is differentiable at a, and find its derivitive.
Since g is differentiable at a, it is also continuous at a. Therefore, since $g(a) \neq 0$, there is a neighborhood around a where g is never 0. For $x \neq a$ in this neighborhood,

$$\begin{aligned} \frac{h(x) - h(a)}{x - a} &= \frac{1}{x - a}\left(\frac{1}{g(x)} - \frac{1}{g(a)}\right) \\ &= \frac{1}{x - a}\left(\frac{g(a) - g(x)}{g(a)g(x)}\right) \\ &= -\frac{g(x) - g(a)}{x - a}\frac{1}{g(a)g(x)}. \end{aligned}$$

Since g is continuous at a and $g(a) \neq 0$, $\lim_{x\to a} \frac{1}{g(a)g(x)} = \frac{1}{g(a)^2}$. Also, since g is differentiable at a, $\lim_{x\to a} \frac{g(x)-g(a)}{x-a}$ exists. Therefore,

$$\lim_{x\to a} \frac{h(x) - h(a)}{x - a} = -\frac{g'(a)}{g(a)^2}$$

is well-defined.

(b) Using part (a) and the product rule, prove the quotient rule: if f, g are differentiable at a, then $\frac{f}{g}$ is also differentiable at a, and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}.$$

Let $h = \frac{1}{g}$, as in part (a), so that $\frac{f}{g} = fh$. Then applying the product rule, followed by the result of part (a) yields:

$$\begin{aligned} (fh)'(a) &= f'(a)h(a) + f(a)h'(a) \\ &= f'(a)\left(\frac{1}{g(a)}\right) + f(a)\left(-\frac{g'(a)}{(g(a))^2}\right) \\ &= \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}. \end{aligned}$$

3.

(a) Using the product rule and induction, show that $(x^n)' = nx^{n-1}$ for all natural n.
First, for the base case, take $n = 1$. The derivative of x at a point a is given by

$$\lim_{x \to a} \frac{x - a}{x - a} = 1,$$

so $x' = 1$, as desired.
Now assume that for some natural number k, $\left(x^k\right)' = kx^{k-1}$. We use the product rule and the base case to take the derivative of x^{k+1}:

$$\begin{aligned} \left(x^{k+1}\right)' &= \left(x^k x\right)' \\ &= \left(x^k\right)' x + x^k x' \\ &= kx^{k-1}x + x^k 1 \\ &= kx^k + x^k \\ &= (k+1)\, x^k \end{aligned}$$

as desired. Therefore, the claim is proved by induction.

(b) Using question 2, find (with proof) $\left(x^{-5}\right)'$.
x^5 has derivative $5x^4$. By question 2(a), $\frac{1}{x^5}$ is differentiable when $x \neq 0$, and

$$\left(x^{-5}\right)' = -\frac{5x^4}{\left(x^5\right)^2} = -\frac{5x^4}{x^{10}} = -5x^{-6}.$$

When $x = 0$, x^{-5} is undefined, and so it is not differentiable there.

4. (28.7) Let $f(x) = x^2$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$.

(a) Sketch the graph of f.

(b) Show that f is differentiable at $x = 0$.
For $x < 0$, $\frac{f(x) - f(0)}{x - 0} = \frac{0}{x} = 0$. Therefore,

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = 0.$$

For $x > 0$, $\frac{f(x) - f(0)}{x - 0} = \frac{x^2}{x} = x$. Therefore,

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} x = 0.$$

Therefore, $\lim_{x \to 0} f(x) = 0$. $f'(0) = 0$.

(c) Calculate f' on $\mathbb{R}$ and sketch its graph.
If $a < 0$, we can find a δ-neighborhood of a, consisting entirely of negative numbers. Take $\delta = -a > 0$, for instance. On this neighborhood, $f \equiv 0$. Restricting our attention to this neighborhood, we can calculate:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{0}{x - a} = 0.$$

If $a > 0$, we can find a δ-neighborhood of a, consisting entirely of positive numbers. On this neighborhood, $f(x) = x^2$. Restricting our attention to this neighborhood, we can calculate:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^2 - a^2}{x - a} = \lim_{x \to a} x + a = 2a.$$

Therefore,

$$f'(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

(Like continuity, differentiability is a local property. Whether or not a function is differentiable at a point is only dependent on the behavior in a small neighborhood around the point.)

(d) Is f' continuous on $\mathbb{R}$? differentiable on $\mathbb{R}$?
f' is continuous on $\mathbb{R}$. Following question 1 in homework 8, f' is gotten by gluing together two continuous function at $x = 0$, and both functions agree at this point, so f' is continuous.
f' is differentiable at any point $a \neq 0$. (f is equivalent to a differentiable function in a neighborhood of these points. See the comment at the end of the last problem.) f' is not differentiable, however, at $a = 0$. To see this, note that

$$\lim_{x \to 0^-} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0^-} \frac{0}{x} = 0$$

while

$$\lim_{x \to 0^+} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0^+} \frac{2x}{x} = 2;$$

so $\lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0}$ is undefined.

5. (28.8) Let

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

(a) Prove that f is continuous at $x = 0$.
Let $\epsilon > 0$. Set $\delta = \sqrt{\epsilon}$. Suppose $|x| < \delta$. If $x \notin \mathbb{Q}$, then $|f(x)| = 0 < \epsilon$. If $x \in \mathbb{Q}$, then $|f(x)| = |x^2| < \delta^2 < \epsilon$. In either case, $|f(x)| < \epsilon$, so f is continuous at 0.

(b) Prove that f is discontinuous at all $x \neq 0$.
As usual, I will prove discontinuity using the sequences definition. Let (r_n) be a sequence of rationals converging to x, and let (s_n) be a sequence of irrationals converging to x. Then $(f(r_n)) = (r_n^2) \to x^2$, while $(f(s_n)) = (0) \to 0$. Since $x^2 \neq 0$, f is discontinuous at x.

(c) Prove that f is differentiable at $x = 0$.
We must prove that $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ exists. We will prove that it equals 0. Let $\epsilon > 0$. Set $\delta = \epsilon$. Assume $0 < |x| < \delta$. If $x \in \mathbb{Q}$, then $\left|\frac{f(x)}{x}\right| = \left|\frac{x^2}{x}\right| = |x| < \delta = \epsilon$. If $x \notin \mathbb{Q}$, then $\left|\frac{f(x)}{x}\right| = \left|\frac{0}{x}\right| = 0 < \epsilon$. This proves our claim.