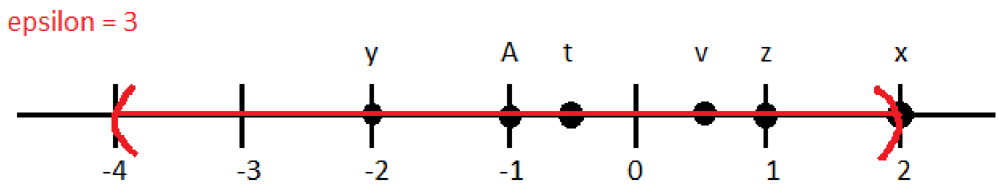
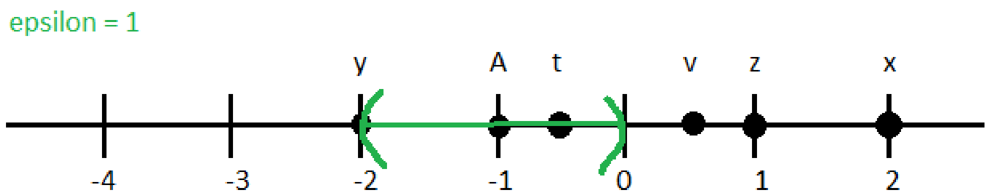
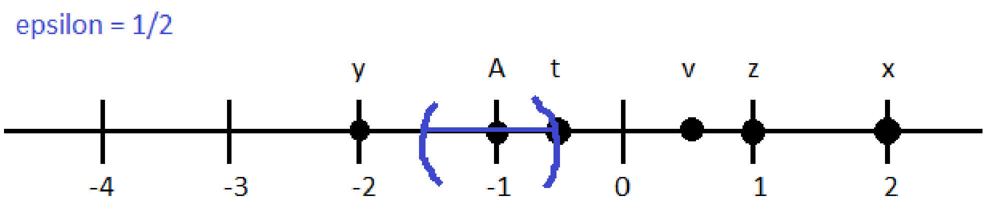
MAT 319, Spring 2012
Solutions to HW 1

1. On the real line, sketch ϵ-neighborhoods of $A = -1$ for $\epsilon = 3$, $\epsilon = 1$, $\epsilon = 1/2$.
For each of these neighborhoods, determine whether it contains points $x = 2$, $y = -2$, $z = 1$, $t = -1/2$, $v = 1/2$.

When $\epsilon = 3$, the ϵ-neighborhood is the open interval $(-4, 2)$, so y, z, t, v are contained in the neighborhood, but x is not.
When $\epsilon = 1$, the ϵ-neighborhood is the open interval $(-2, 0)$, so t is contained in the neighborhood while x, y, z, v are not.
When $\epsilon = 1/2$, the ϵ-neighborhood is the open interval $(-3/2, -1/2)$, so none of the points are contained.







2. Consider the sequence (s_n), where
$$s_n = \begin{cases} n & \text{if } n \leq 100 \\ 2 & \text{if } n > 100. \end{cases}$$
Does this sequence converge? Prove your answer.

This sequence converges to 2. There are two ways to prove it.

1) We have to show that every neighborhood of $A = 2$ contains a tail of (s_n). But since every neighborhood contains its center, the point $A = 2$ itself, it is clear that every neighborhood will contain the tail $(s_{101}, s_{102}, s_{103}, \dots)$.

2) For an epsilon-proof, let $\epsilon > 0$ be arbitrary given number. Then for all $n > 100$, $s_n = 2$, so in particular, $|s_n - 2| = 0 < \epsilon$.

3. Determine whether the following sequences converge. If so, find the limits. Prove your answers.

(a) the sequence (x_n), where $x_n = \frac{1}{n+17}$.

We prove that the sequence converges, and its limit is 0. This can be shown in several slightly different ways.
1) For a neighborhoods-and-tails argument, we notice that our sequence

$$\frac{1}{18}, \frac{1}{19}, \frac{1}{20}, \frac{1}{21}, \frac{1}{22}, \frac{1}{23}, \dots$$

looks like the familiar $(\frac{1}{n})$ sequence, with a few first terms thrown away. We know that $(\frac{1}{n})$ converges to 0, in other words, that every neighborhood of 0 contains a tail of $(\frac{1}{n})$. But this guarantees that the "corresponding" tail of (x_n) also lies in the given neighborhood.
2) For an epsilon-proof, let $\epsilon > 0$. To establish convergence, we must find N such that $|x_n| < \epsilon$ for all $n > N$. Let $N = \dfrac{1}{\epsilon} - 17$. Then if $n > N$,

$$|x_n| = \frac{1}{n+17} < \frac{1}{N+17} = \frac{1}{\frac{1}{\epsilon} - 17 + 17} = \epsilon.$$

This concludes the proof. As a side note, we could just as easily have taken N to be anything larger than this, in particular, $N = \frac{1}{\epsilon}$. With this choice, the computation reads:

$$|x_n| = \frac{1}{n+17} < \frac{1}{n} < \frac{1}{N} = \epsilon.$$

(b) the sequence (y_n), where

$$y_n = \begin{cases} 0 & \text{if } n = 17 \\ \frac{1}{n-17} & \text{otherwise} \end{cases}$$

This sequence also converges, with limit 0.
1) For a neighborhoods-and-tails argument, notice that even though the few first terms in our sequence look "weird", after a while we are just getting the familiar sequence $(\frac{1}{n})$:

$$-\frac{1}{16}, -\frac{1}{15}, \dots, 0, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$$

Since any given neighborhood of 0 contains a tail of $(\frac{1}{n})$ (because we know that $(\frac{1}{n})$ converges to 0), we can find the corresponding tail of (y_n) contained in the given neighborhood.
2) Epsilon-proof: let $\epsilon > 0$. Let $N = \frac{1}{\epsilon} + 17 > 17$. Then if $n > N$,

$$|y_n| = \frac{1}{n-17} < \frac{1}{N-17} = \epsilon,$$

as desired. Note that we could not take $N = \frac{1}{\epsilon}$ because $\dfrac{1}{n-17} > \dfrac{1}{n}$.

4. Suppose that the sequence (x_n) converges to 0. Prove that the sequence

$$x_1, 0, x_2, 0, x_3, 0, x_4, 0, \dots,$$

constructed by alternating the terms of the sequence (x_n) with zeroes, also has limit 0.

1) Here, a neighborhoods-and-tails proof is easier. Given a neighborhood of 0, we must show that it captures some tail of our sequence. But since $\lim x_n = 0$, there is a tail of (x_n) contained in the given neighborhood. Also, 0 is always contained in any neighborhood of 0. Thus, once the required

tail of (x_n) starts, we can be sure that the corresponding tail of our sequence is also in the given neighborhood.

2) Finding N for epsilon requires more formulas. The sequence in question can be defined by

$$a_n = \begin{cases} x_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Let $\epsilon > 0$. Since (x_n) converges to 0, there exists a number N such that if $n > N$, then $|x_n| < \epsilon$. Now suppose $n > 2N$, so that $\frac{n+1}{2} > N$.

If n is even, then $|a_n| = 0 < \epsilon$. If n is odd, then $|a_n| = \left|x_{\frac{n+1}{2}}\right| < \epsilon$. Thus, $|a_n| < \epsilon$ for all $n > N$. Hence, (a_n) converges to 0.

5. Let (x_n) and (y_n) be two sequences converging to 0. Consider a new sequence where x_n's and y_n's alternate:

$$x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, \ldots$$

Prove that this sequence converges to 0.

1) Neighborhoods-and-tails are probably easier here, too. Given any neighborhood of 0, we can find a tail of (x_n) contained in that neighborhood and a tail of (y_n) contained in that neighborhood. The problem is that we would like to put those tails together, but they might start at different places, if we have an N-tail for (x_n) and an M-tail for (y_n). To be in the given neighborhood, we have to make sure that *both* tails have started. For this, just take a tail where the indices for both (x_n) and (y_n) are greater than *the larger of M and N*.

2) If you want to an epsilon-proof, the sequence in question can be defined by

$$a_n = \begin{cases} x_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ y_{n/2} & \text{if } n \text{ is even.} \end{cases}$$

Let $\epsilon > 0$. Since (x_k) converges to 0, there exists N_x such that if $k > N_x$, then $|x_k| < \epsilon$. Similarly, we can find N_y such that if $k > N_y$, then $|y_k| < \epsilon$. Let $N = 2\max\{N_x, N_y\}$, and suppose $n > N$. If n is odd, then $\frac{n+1}{2} > \frac{N+1}{2} > \frac{2N_x}{2} = N_x$, so $|a_n| = \left|x_{\frac{n+1}{2}}\right| < \epsilon$. If n is even, then $\frac{n}{2} > N_y$, so $|a_n| = \left|y_{n/2}\right| < \epsilon$. Thus, in either case, $|a_n| < \epsilon$. Therefore, the N-tail is entirely contained within the ϵ-neighborhood of 0, and (a_n) converges to 0.