MAT 319, Spring 2012
Solutions to Midterm 1

1. Suppose that the sequence x_n is bounded and the sequence y_n diverges to $+\infty$.

   (a) Show that there is some N such that $y_n \neq 0$ for every $n > N$,
   y_n diverges to $+\infty$, so by definition, for every α, there exists an N such that if $n > N$, then $y_n > \alpha$. In particular, we can take $\alpha = 0$: there exists some number N such that for all $n > N$, $y_n > 0$. Thus, also $y_n \neq 0$ for $n > N$.

   (b) Prove that the sequence $\left(\frac{x_n}{y_n}\right)$ converges to 0.
   (x_n) is bounded, so there exists some number $M > 0$ such that $|x_n| < M$ for all n. Let $\epsilon > 0$. Since y_n diverges to $+\infty$, we can find N such that if $n > N$, then $y_n > \frac{M}{\epsilon}$. Therefore, for $n > N$,

$$\left|\frac{x_n}{y_n}\right| < \frac{M}{M/\epsilon} = \epsilon.$$

   Hence, (x_n/y_n) converges to 0, as desired.

2. Let (x_n) be a sequence such that $x_n > 0$ for all n. Suppose that (x_n) converges to 0.

   (a) What is $\inf(x_n)$? Prove your answer.
   $\inf(x_n) = 0$. First, since $x_n > 0$ for all n, 0 is indeed a lower bound. We must show it is a greatest lower bound. Let $\epsilon > 0$ be arbitrary. We will show that ϵ is not a lower bound. Since x_n converges to 0, we can find a number N such that if $n > N$, then $|x_n| < \epsilon$. In particular, we must have $x_n < \epsilon$ for all $n > N$. Therefore, ϵ is not a lower bound, and 0 is the greatest lower bound.

   (b) Prove that (x_n) cannot have an increasing subsequence.
   Suppose that (x_{n_k}) were an increasing subsequence. Since (x_n) converges to 0, (x_{n_k}) must converge to 0, as well. x_{n_1} is positive, so (treating it as ϵ) we can find K such that if $k > K$, then $x_{n_k} = |x_{n_k}| < x_{n_1}$. But $x_{n_1} \leq x_{n_k}$ for all k since the subsequence is increasing. Contradiction. Therefore, (x_n) is not an increasing subsequence.

3. Let $\sum a_n$ be a series with all positive terms, $a_n > 0$ for all n.

   (a) Suppose that

$$\sqrt[n]{a_n} < \frac{1}{2} \text{ for every } n.$$

   Prove that $\sum a_n$ converges.
   We have that $0 < a_n < \left(\frac{1}{2}\right)^n$ for all n. $\sum \left(\frac{1}{2}\right)^n$ is a convergent geometric series; therefore, by the comparison test, $\sum a_n$ converges as well.

   (b) Suppose that

$$\lim \sqrt[n]{a_n} = \frac{1}{4}.$$

   Prove that $\sum a_n$ converges.
   Since $\sqrt[n]{a_n}$ converges to $\frac{1}{4} < \frac{1}{2}$, there exists some N such that if $n > N$, then $\sqrt[n]{a_n} < \frac{1}{2}$. For $n > N$, we have $0 < a_n < \left(\frac{1}{2}\right)^n$, and therefore, by the comparison test,

$$\sum_{n=N+1}^{\infty} a_n \leq \sum_{n=N+1}^{\infty} \left(\frac{1}{2}\right)^n$$

   converges. (The right-hand sum is a convergent geometric series.) Thus,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

   converges, since we are only adding on finitely many terms to a convergent series.

4. Determine whether the following sequences converge or diverge. In case of convergence, find the limit.

   (a) $\dfrac{n-1}{\sqrt{n^4+n+1}}$.
   Since $n \geq 1$, we can divide the top and bottom by n^2 to get the equivalent expression:

$$\frac{\frac{1}{n}-\frac{1}{n^2}}{\sqrt{1+\frac{1}{n^3}+\frac{1}{n^4}}}.$$

   Since $\frac{1}{n} \to 0$, the limit product formula implies that $\frac{1}{n^k} \to 0$ for any $k \in \mathbb{N}$. Then the limit sum and difference rules tell us that

$$\frac{1}{n}-\frac{1}{n^2} \to 0$$

   and

$$1+\frac{1}{n^3}+\frac{1}{n^4} \to 1.$$

   By the limit root formula,

$$\sqrt{1+\frac{1}{n^3}+\frac{1}{n^4}} \to 1.$$

   Since the denominator of the overall expression does not tend to 0, we can apply the limit quotient formula to get

$$\frac{\frac{1}{n}-\frac{1}{n^2}}{\sqrt{1+\frac{1}{n^3}+\frac{1}{n^4}}} \to \frac{0}{1} = 0.$$

   (b) $\dfrac{(-1)^n n}{n+1}$.
   This sequence diverges. Take the subsequence of even terms $(x_{2k}) = \frac{2k}{2k+1} = \dfrac{1}{1+(2k)^{-1}}$. $(2k)^{-1} \to 0$ as $k \to \infty$, so using the arithmetic limit laws, $x_{2k} \to 1$.
   Similarly, look at the subsequence of odd terms $(x_{2k-1}) = \frac{-(2k-1)}{(2k-1)+1} = \dfrac{-1}{1+(2k-1)^{-1}}$. This converges to -1 as $k \to \infty$. (Students should fill in the details of these claims.)
   Therefore, we have found two subsequences that converge to different limits. Thus, the sequence cannot converge.