MAT 311 Introduction to Number Theory

Problem Set 9

Solutions

Problem 1 sec 7.1 Just divide, reversing the fractions and applying the division algorithm as you go along:

$$\frac{17}{3} = 5 + \frac{1}{3/2} = 5 + \frac{1}{1 + 1/2};$$

other numbers are even easier.

Problem 3 sec 7.1 Just compute.

Problem 5 sec 7.1 Using induction by n, prove that $\langle a_0, a_1, \dots a_n \rangle > \langle a_0, a_1, \dots a_n + c \rangle$ when n is odd, but $\langle a_0, a_1, \dots a_n \rangle < \langle a_0, a_1, \dots a_n + c \rangle$ when n is even. Check the base of induction for $n = 1$ and $n = 2$. For the induction step, increase n by 2 (to keep the same parity). Use the fact that

$$\langle a_0, a_1, \dots a_{n+2} \rangle = a_0 + \frac{1}{a_1 + \frac{1}{\langle a_2, a_3, \dots a_{n+2} \rangle}} \text{ and } \langle a_0, a_1, \dots a_{n+2} + c \rangle = a_0 + \frac{1}{a_1 + \frac{1}{\langle a_2, a_3, \dots a_{n+2} + c \rangle}};$$

since $\langle a_2, a_3, \dots a_{n+2} + c \rangle$ and $\langle a_2, a_3, \dots a_{n+2} + c \rangle$ contain two terms fewer, the induction hypothesis applies, and the first expression is less then the second if n is odd (or greater if n is even). For the case n odd, we then see that

$$a_1 + \frac{1}{\langle a_2, a_3, \dots a_{n+2} \rangle} < a_1 + \langle a_2, a_3, \dots a_{n+2} + c \rangle,$$

and

$$a_0 + \frac{1}{a_1 + \frac{1}{\langle a_2, a_3, \dots a_{n+2} \rangle}} > a_0 + \frac{1}{a_1 + \frac{1}{\langle a_2, a_3, \dots a_{n+2} + c \rangle}}.$$

For n even the inequalities work the other way round.

Problem 3 sec 7.3 (b) The value of an infinite continued fraction $\langle a_0, a_1, a_2, \dots \rangle$ is equal to

$$\xi = \lim \langle a_0, a_1, \dots a_n \rangle = \lim \langle a_0, a_1, \dots a_{n+2} \rangle = a_0 + \frac{1}{a_1 + \frac{1}{\lim \langle a_2, a_3, \dots a_n \rangle}}.$$

From the periodic pattern in the fraction, it follows that

$$\xi = 1 + \frac{1}{2 + \frac{1}{\xi}}.$$

This gives the quadratic equation $2\xi + 2 = \xi(2\xi + 1)$, or $2\xi^2 - 2\xi - 1 = 0$. Solving at taking the positive root, we find that $\xi = \frac{1+\sqrt{3}}{2}$.

Part (c) is similar, (a) is easier; (c) and (d) can be reduced to (b), since, for example,

$$\langle 1, 3, 1, 2, 1, 2, \dots \rangle = 1 + \frac{1}{3 + \frac{1}{\langle 1, 2, 1, 2, \dots \rangle}}$$

Binary expansions question (a) The value of a binary number $.a_1a_2a_3\ldots$ with digits $a_1, a_2, \ldots$ that are all equal to 0 or 1 is the infinite sum

$$\frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \frac{a_4}{2^4} + \ldots.$$

(check that this gives a finite answer!) To construct the binary expansion of a number $0 < \xi < 1$, set $a_1 = 0$ if $\xi < 1/2$, $a_1 = 1$ if $\xi \geq 1/2$, then $a_2 = 0$ if $\xi - .a_1 < 1/2$, $a_2 = 1$ if $\xi - .a_1 \geq 1/2$, then $a_1 = 0$ or 1 depending on $\xi - .a_1a_2$, and so on. (We ignore the issue of non-unique expansions of the sort $1/2 = .01111111... = 0.1$, cf $0.39999999 = 0.4$ for decimals.)

(b) The value of a finite binary fraction $.a_1a_2a_3\ldots a_n$ is

$$\frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \ldots \frac{a_n}{2^n} = \frac{a_1\, 2^{n-1} + a_2\, 2^{n-2} + \cdots + a_n}{2^n},$$

which is a rational number whose denominator (in lowest terms) is a power of 2. Conversely, given a rational number of the form $\frac{p}{2^n}$, it can be written as a finite expansion whose n digits are just the binary digits of p.

(c) The answer is the same as for the decimal case: the numbers with periodic expansions are exactly the rationals. To see this, write

$$\begin{aligned} .b_1\ldots b_m a_1a_2\ldots a_k a_1a_2\ldots a_k a_1a_2\ldots a_k\cdots = & \frac{b_1\, 2^{n-1} + b_2\, 2^{m-2} + \cdots + b_m}{2^m} \\ & + \frac{a_1\, 2^{k-1} + a_2\, 2^{k-2} + \cdots + a_k}{2^{k+m}} + \frac{a_1\, 2^{k-1} + a_2\, 2^{n-2} + \cdots + a_k}{2^{2k+m}} + \ldots, \end{aligned}$$

and simplify

$$\begin{aligned} & \frac{a_1\, 2^{k-1} + a_2\, 2^{k-2} + \cdots + a_k}{2^{k+m}} + \frac{a_1\, 2^{k-1} + a_2\, 2^{n-2} + \cdots + a_k}{2^{2k+m}} + \cdots = \\ & \left(a_1\, 2^{k-1} + a_2\, 2^{k-2} + \cdots + a_k\right)\left(\frac{1}{2^{k+m}} + \frac{1}{2^{3k+m}} + \frac{1}{2^{3k+m}} + \ldots\right) \end{aligned}$$

by summing the geometric series.

For the converse (showing that every rational has a periodic expansion), observe that in the expansion process for p/q, we repeatedly (for increasing n) divide $2^n p$ by q and then continue with the remainders. Because there are only finitely many remainders mod q, one of the remainders will at some point repeat, and then the process will become periodic.

Details can be found (at least for the decimal case, which is very similar) in many analysis textbooks (for example Apostol). This material is not included in the final exam.