MAT 311 Introduction to Number Theory

Problem Set 10

Solutions

Problem 1 sec 7.4 Let's expand $\sqrt{2}$.

$$\sqrt{2} = 1+(\sqrt{2}-1) = 1+\cfrac{1}{\cfrac{1}{\sqrt{2}-1}} = 1+\frac{1}{1+\sqrt{2}} = 1+\frac{1}{2+(\sqrt{2}-1)} = 1+\cfrac{1}{2+\cfrac{1}{1+\sqrt{2}}} = \ldots,$$

which gives the answer $\sqrt{2} = \langle 1, 2, 2, 2, 2, \ldots \rangle$ since the pattern repeats.

Expansions of other numbers in this question also have repeating patterns.

Problem 2 sec 7.4 Recall that

$$\frac{h_n}{k_n} = \langle a_0, a_1, \ldots a_n \rangle.$$

In particular, $\dfrac{h_0}{k_0} = a_0$, so if the 0-th convergents are the same, then the a_0 terms are the same. Next,

$$\frac{h_1}{k_1} = a_0 + \frac{1}{a_1}$$

and, if these are the same, then the a_1 terms are the same because we already know that a_0's are the same. Continue in this manner by induction to show that if $a_0, a_1, \ldots a_m$ terms are the same and the convergents $\frac{h_{m+1}}{k_{m+1}}$ are the same, then the a_{m+1} terms are the same.

Problem 3 sec 7.4 Using the result of Problem 2, we know that the coefficients of the continued fraction expansions of α and γ are the same up to the a_n term; we need to show that β also has the same coefficients up to the n-th term. Since the a_0 term in the expansion of ξ is the greatest integer not greater than ξ, $\alpha < \beta < \gamma$, and this number is the same for α and γ, β has the same a_0 term. For the next coefficient, recall that a_1 is the greatest integer not greater than $\dfrac{1}{\xi - a_0}$, and use the same reasoning. Continue by induction up to the nth term.

Problem 3 sec 7.5 We need to show that

$$|\xi k_n - h_n| = \min_{\text{all } x, 0<y\leq b} |\xi y - x|.$$

But this follows from thm 7.13, since we can only have $|\xi y - x| < |\xi k_n - h_n|$ for $y > k_{n+1}$.

Problem 4 sec 7.5 Suppose $\frac{a}{b}$ is a good approximation, and find two terms of the (increasing) sequence k_i, such that $k_n \leq b < k_{n+1}$. By thm 7.13 we can't have , because otherwise b would be $\geq k_{n+1}$. On the other hand, because $|\xi b - a| = \min$ and $k_n \geq b$, we can't have $|\xi k_n - h_n| < |\xi b - a|$. So it must be that $|\xi k_n - h_n| = |\xi b - a|$. Now examine the proof of thm 7.13, introduce x and y, and follow all the steps of the proof to see that the equality $|\xi k_n - h_n| = |\xi b - a|$ can only hold if $y = 0$, $x = 1$. But that means $a = h_k$, $b = k_n$.

Problem 1 sec 7.6 We know that there's at least one rational number like this among any three consecutive convergents for $\sqrt{2}$. Write out the continued fraction and check the first few convergents; actually the very first two give the answer.