MAT 311 Introduction to Number Theory

## Problem Set 4

due Wednesday, February 25

Please prove all your answers.

**Problem 1.** Solve the following systems of congruences.

(a)
$$\begin{aligned} x &\equiv 3 \mod 5 \\ x &\equiv 2 \mod 8 \\ x &\equiv 0 \mod 7 \end{aligned}$$

(b)
$$\begin{aligned} 13x &\equiv 2 \mod 15 \\ 16x &\equiv 3 \mod 25 \end{aligned}$$

(c)
$$\begin{aligned} x &\equiv 0 \mod 18 \\ 3x &\equiv 12 \mod 20 \\ 2x &\equiv -2 \mod 30 \end{aligned}$$

**Problem 2.** Prove that $7|(3^{2n+1} + 2^{n+2})$ for all $n$.

**Problem 3.** For what $n$ is $\phi(n)$ odd?

**Problem 4.** Prove that

$$(p-1)! \equiv p-1 \mod (1+2+3+\cdots+(p-1)) \text{ if } p \text{ is prime.}$$

**Hint:** simplify $1+2+3+\cdots+(p-1)$ first. Then think of some familiar congruence involving $(p-1)!$.

**Problem 5.** (a) Find the last digit of $2^{1000}$ and the last digit of $3^{1000}$.

(b) Find the last two digits of $3^{1000}$. **Hint:** use a theorem about $a^n \equiv ? \mod$ something.

(c) Find the last two digits of $2^{1000}$. **Hint:** the theorem you used in (b) no longer applies directly, but you can work around this by factoring $100 = 4 \cdot 25$ and dealing with mod 4 and mod 25 separately.