MAT 311 Introduction to Number Theory

Problem Set 2

due Wednesday, February 11

Please prove all your answers.

Questions 1 and 2 don't have much to do with number theory, they are just for extra practice with induction. Use "regular" or complete induction as appropriate.

Problem 1. Find all natural numbers n such that $n^2 > 2^n$. Prove your answer.

Problem 2. Let $x > 0$ be such that $x + \dfrac{1}{x}$ is an integer (x itself doesn't have to be an integer). Prove that $x^n + \dfrac{1}{x^n}$ is an integer for every natural n.

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Problem 3. Prove that there are infinitely many primes of the form $p = 4k + 3$. **Hint:** Show that every number of the form $4k + 3$ must have a prime divisor of the same form, and mimic Euclid's proof of infinitude of primes.

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Problem 4. The following identities will be needed to solve Questions 5 and 6 (**hint hint**). Prove them by any method.
(a) $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1})$ for all natural n.
(b) $x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots - xy^{n-2} + y^{n-1})$ for all n odd.

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Problem 5. Suppose that $2^n - 1$ is a prime number. Show that n must be also prime.

Problem 6. Suppose that $2^n + 1$ is a prime number. Show that $n = 2^k$ for some k.
(Prime numbers of the form $2^{2^k} + 1$ are called Fermat's primes. Since $2^{2^k} + 1$ is prime for k=1,2,3,4, Fermat conjectured that all numbers $2^{2^k} + 1$ are prime. However $2^{32} + 1$ is composite.)

Problem 7. Read about Eratosthene's sieve somewhere (eg in exercise 25 p. 30 of the textbook). Use this method to find all primes that are less than 300. Please explain what exactly you are doing, and why this works.