MAT 311, Homework 2 due 9/13

1. (a) Show that there are no prime triplets,that is, primes p, $p+2$, $p+4$, other than 3, 5, and 7.

(b) Find all prime p such that p^2+2 is also prime.

2. (a) Suppose that 2^n-1 is a prime number. Show that n must be also prime.

(b) Prove that n^4+n^2+1 is composite for all natural $n>1$. **Hint:** $a^2+a+1=(a+1)^2-a$.

(c) Find the prime decomposition of $2^{36}-1$. Don't use a calculator or an answer from the internet, and don't compute the decimal representation of $2^{36}-1$. Use algebra instead; parts (a) and (b) should be useful, among other things.

3. Prove the following criterion for divisibility by 11:

A natural number is congruent modulo 11 to the alternating sum of its digits (in decimal notation).

"Alternating" means taken with alternating signs, + for the units, − for tens, + for hundreds, etc. (Example: $123456 \equiv -1+2-3+4-5+6 \mod 11$.)

4. (a) Prove that a square of an integer cannot end by two odd digits (in decimal notation).

(b) For n integer, prove that if the last digit of n^2 is 5, then n^2 ends by 25 (in decimal notation).

(c) Let $m = 101010....10101$ be the number that consists of 2024 1's alternating with 2023 0's (in decimal notation. Show that m is not square of any integer.

(d) Suppose the integer a in decimal notation consists of 300 1's together with some number of zeros. Show that a is not a square of any integer.

5. (a) Find the remainder of $(116+17^{17})^{21}\cdot 7^{49}$ when divided by 8.

(b) Show that $6^{2n+1}+1$ is divisible by 7 for all $n \geq 1$.

(c) Show that 2^n-3 divides $(2^n-1)^n-3$ for every $n \geq 1$.

Please use congruences, do not use induction in (b) and (c)!

6. Suppose that $u=am+bn$, $v=cm+dn$, and $ad-bc=1$ for some integers a,b,c,d,m,n,u,v. Prove that $(m,n)=(u,v)$.