Problem: Find a Taylor series centered at the given value of a.

(I) $f(x) = \cos x \qquad a = \frac{\pi}{2}$

There are two ways to solve:

1 way: To use one of the ready Maclauren series and a formula to reduce your function to the template

2 way: To calculate the coefficients from the derivatives using

$$f(x) = \sum_{n=0}^{\infty} d_n \cdot (x-a)^n$$

$$d_n = \frac{f^{(n)}(a)}{n!}$$

1 way: $f(x) = \cos x$, and we need to see powers of $(x - \frac{\pi}{2})$

$\Rightarrow$ ~~[illegible]~~ We have $x = \underbrace{\left(x - \frac{\pi}{2}\right)}_{B} + \frac{\pi}{2} = B + \frac{\pi}{2}$

$\rightarrow \cos x = \cos\left(B + \frac{\pi}{2}\right)$

Need a formula to write it using only $\cos B$ and $\sin B$ (nothing else inside)

There is: The shift formula:

$$\cos\left(B+\frac{\pi}{2}\right) = -\sin B$$

Now we know $\sin B = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot B^{2n+1}}{(2n+1)!}$

$$\Rightarrow \cos\left(B+\frac{\pi}{2}\right) = -\sin B = -\sum_{n=0}^{\infty} \frac{(-1)^n \cdot B^{2n+1}}{(2n+1)!}$$

$$\cos\left(B+\frac{\pi}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \cdot B^{2n+1}}{(2n+1)!}$$

$$\Rightarrow \boxed{\cos x = \cos\left(\underbrace{\left(x-\frac{\pi}{2}\right)}_{B} + \frac{\pi}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \cdot \left(x-\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}}$$

2way: Need to find all the derivatives: $d_n = \frac{f^{(n)}\left(\frac{\pi}{2}\right)}{n!}$

$f(x) = \cos x$ $\qquad d_0 = f\left(\frac{\pi}{2}\right) = 0$

$f'(x) = (\cos x)' = -\sin x$ $\qquad d_1 = \frac{f'\left(\frac{\pi}{2}\right)}{1!} = \frac{-1}{1!}$

$f''(x) = (-\sin x)' = -\cos x$ $\qquad d_2 = \frac{f''\left(\frac{\pi}{2}\right)}{2!} = 0$

$f'''(x) = (-\cos x)' = \sin x$ $\qquad d_3 = \frac{f'''\left(\frac{\pi}{2}\right)}{3!} = \frac{1}{3!}$

$f^{IV}(x) = (\sin x)' = \cos x$ $\qquad d_4 = 0$

So we loop after f^{IV}

$\Rightarrow$ For even k $\quad d_k = 0$

For odd k $\quad d_k = \frac{(-1)^?}{k!}$

When k - odd $\Rightarrow k = 2n+1$

and we see that $d_{2n+1} = \frac{(-1)^{n+1}}{(2n+1)!}$

(Because when $n = 0 \to k = 1$ and we know $d_k = \frac{(-1)^{1 \to 0+1}}{1!}$)

$\Rightarrow$ We will have to split our sum into two parts

$$\sum_{k=0}^{\infty}(\ ) = \sum_{k=\text{even}}(\ ) + \sum_{k=\text{odd}}(\ )$$

For the even k $\quad d_k = 0$

so for the even part $\Big\} \Rightarrow \sum_{k=\text{even}} d_k \cdot \left(x - \frac{\pi}{2}\right)^k = \sum_{k=\text{even}} 0 = 0$

$$\Rightarrow \quad \cos x = \sum_{k=\text{odd}} d_k \cdot \left(x - \frac{\pi}{2}\right)^k = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \cdot \left(x - \frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}$$

(remember that $k = 2n+1$
$k = \text{odd} \to n = 0, 1, 2, \dots, \infty$)

So we get the same answer.

The first way is usually shorter.

(II) $f(x) = \ln x$ $a = 3$

I way: We want to see powers of $(x-3)$

$$x = \underbrace{(x-3)}_{B} + 3 = (B+3)$$

$$\to \ln x = \ln(B+3)$$

Our template is for $\ln(1-A) = -\sum_{n=1}^{\infty} \frac{A^n}{n}$

Need to see $1+$ something

$$\ln(B+3) = \ln(3+B) = \ln\left(3\cdot\left(1+\frac{B}{3}\right)\right)$$

Now we need to split the 3 from $\left(1+\frac{B}{3}\right)$

We have a formula

$$\ln(a\cdot b) = \ln a + \ln b$$

$$\Rightarrow \ln(B+3) = \ln 3 + \ln\left(1+\frac{B}{3}\right)$$

The $\ln 3$ part is good, it is just a number (so we do nothing about it)

The $\ln\left(1+\frac{B}{3}\right)$ part is like the template $\ln(1-A)$

$$1 + \frac{B}{3} \quad \Rightarrow A = -\frac{B}{3}$$

$$\ln\left(1+\frac{B}{3}\right) = \ln\left(1-\left(-\frac{B}{3}\right)\right) = -\sum_{n=1}^{\infty} \frac{\left(-\frac{B}{3}\right)^n}{n} =$$

$$= -\sum_{n=1}^{\infty} \frac{(-1)^n . B^n}{3^n . n}$$

$$\Rightarrow \ln\left(1+\frac{B}{3}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} . B^n}{n . 3^n}$$

$$\Rightarrow \ln(B+3) = \ln 3 + \ln\left(1+\frac{B}{3}\right) =$$

$$= \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} . B^n}{n . 3^n}$$

$B = (x-3)$

$$\Rightarrow \boxed{\ln x = \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} . (x-3)^n}{n . 3^n}}$$

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2 way: $\quad f(x) = \ln x$

Need ~~the~~ coefficients d_n for

$$\sum_{n=0}^{\infty} d_n . (x-3)^n$$

$$d_n = \frac{f^{(n)}(3)}{n!}$$

$$d_0 = f(3) = \ln 3$$

$$f'(x) = \frac{1}{x} = x^{-1} \Rightarrow d_1 = \frac{f'(3)}{1!} = \frac{1}{1.3}$$

$$f''(x) = (x^{-1})' = -1 \cdot x^{-2} \Rightarrow d_2 = \frac{f''(3)}{2!} = \frac{-1}{2 \cdot 3^2}$$

$$f'''(x) = -1 \cdot -2 \cdot x^{-3} = (-1)^2 \cdot 1.2 \cdot x^{-3}$$

$$f'''(3) = (-1)^2 \cdot 2! \, 3^{-2}$$

$$f'''(3) = \frac{(-1)^2 \cdot 2!}{3^2}$$

$$d_3 = \frac{f'''(3)}{3!} = \frac{(-1)^2 \cdot \cancel{1.2}}{\cancel{1.2}.3 \cdot 3^2}$$

$$d_3 = \frac{(-1)^2}{3.3^2}$$

$$f^{IV}(x) = \left((-1)^2 \cdot 1.2 \cdot x^{-3}\right)' =$$

$$(-1)^2 \cdot 1.2 \cdot (-3) \cdot x^{-4} =$$

$$= (-1)^3 \cdot 3! \; x^{-4} = \frac{(-1)^3 \cdot 3!}{x^4}$$

$$\Rightarrow d_4 = \frac{(-1)^3 \cdot \cancel{3!}}{\cancel{4!}_{\,4} \, 3^4}$$

$$d_4 = \frac{(-1)^3}{4 \cdot 3^4}$$

$$f^{IV}(3) = \frac{(-1)^3 \cdot 3!}{3^4}$$

$\Rightarrow$ We have a pattern for $f', \ldots, f^{(n)}$:

$$f^{(n)}(x) = (-1)^{n-1} \cdot (n-1)! \, x^{-n}$$

$$\Rightarrow d_n = \frac{f^{(n)}(3)}{n!} = \frac{(-1)^{n-1} \cdot \cancel{(n-1)!} \, 3^{-n}}{\cancel{n!}_{\,n}} =$$

$$= \frac{(-1)^{n-1}}{n \cdot 3^n}$$

$$\Rightarrow \boxed{d_n = \frac{(-1)^{n-1}}{n \cdot 3^n}}$$ but for $n \geq 1$

The pattern is different for $n=0$
$(d_0 = \ln 3)$

$\Rightarrow$ need to split:

$$\sum_{n=0}^{\infty}() = \left(\begin{matrix}\text{the} \\ n=0 \\ \text{term}\end{matrix}\right) + \sum_{n=1}^{\infty}()$$ (the $n=0$ term is just d_0!)

$$\Rightarrow \boxed{\ln x = \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot (x-3)^n}{n \cdot 3^n}}$$

Now notice that these answers on first look differ:

one has $(-1)^{n+1}$, the other $(-1)^{n-1}$

But since $(-1)^{n+1} = (-1)^{n-1} \cdot (-1)^2 = (-1)^{n-1} \cdot 1$

the two answers are the same.

III. $f(x) = e^x$ at $a = 4$

1 way: Need to see powers of $(x-4)$

$$x = \underbrace{(x-4)}_{B} + 4 = B + 4$$

$\Rightarrow e^x = e^{B+4}$, we have a formula:

$$e^{B+4} = e^B \cdot e^4 = e^4 \cdot e^B$$

And e^4 is just a number, so we leave it.

$$e^B = \sum_{n=0}^{\infty} \frac{B^n}{n!} \Rightarrow e^{(x-4)} = \sum_{n=0}^{\infty} \frac{(x-4)^n}{n!}$$

$$\Rightarrow e^x = e^4 \cdot e^B = e^4 \cdot e^{(x-4)} = \boxed{e^4 \cdot \sum_{n=0}^{\infty} \frac{(x-4)^n}{n!}}$$

$$\Rightarrow \boxed{e^x = \sum_{n=0}^{\infty} \frac{e^4 \cdot (x-4)^n}{n!}}$$

can leave it like this if you want

2 way is always longer, so I will not do it.

Last example: $f(x) = \cos x$ at $a = \pi$

$$x = (x-\pi) + \pi = (B + \pi)$$

use template

$$\cos(B+\pi) = \left(\begin{matrix}\text{need}\\ \text{formula}\end{matrix}\right) = -\cos B = -\sum_{n=0}^{\infty} \frac{(-1)^n \cdot B^{2n}}{(2n)!}$$

$\rightarrow$ shift formula

clean up:

$$\Rightarrow \cos x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \cdot (x-\pi)^{2n}}{(2n)!}$$

(moved the $-$ inside the Σ, and $B = (x-\pi)$)