Tritangent circles to a generic curve

September 22, 2015

Pre-history

For a circle generically immersed to the plane

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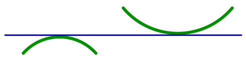
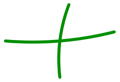
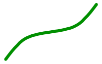
$\#($  $)$ $-$ $\#($  $)$ $=$ $\#($  $)$ $+$ $\frac{1}{2}\#($  $)$

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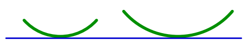
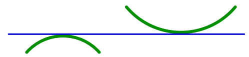
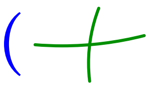
$\#($  $)$ $-$ $\#($  $)$ $=$ $\#($  $)$ $+$ $\frac{1}{2}\#($  $)$

Fabricius-Bjerre formula (1962):

$$e - i = n + \frac{1}{2} f$$

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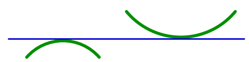
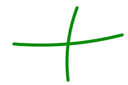
$\#($  $) - \#($  $) = \#($  $) + \frac{1}{2}\#($  $)$

Fabricius-Bjerre formula (1962):

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Ferrand splitted this formula (1997):

$$\begin{aligned} e^+ - i^+ &= \ \ J^+ + w^2 - 1 + \frac{1}{2}f \\ e^- - i^- &= -J^- - w^2 + 1, \end{aligned}$$

where w is the Whitney number and $J^\pm$ are the Arnold invariants.

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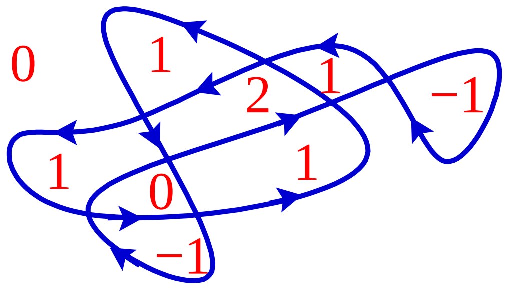
In pictograms:

$$\#(\quad) - \#(\quad) = \ \ J^+ + w^2 - 1 + \frac{1}{2}\#(\quad)$$

$$\#(\quad) - \#(\quad) + \#(\quad) - \#(\quad) = -J^- - w^2 + 1$$

Winding numbers

Winding numbers of faces: $f \mapsto i_f \in \mathbb{Z}$



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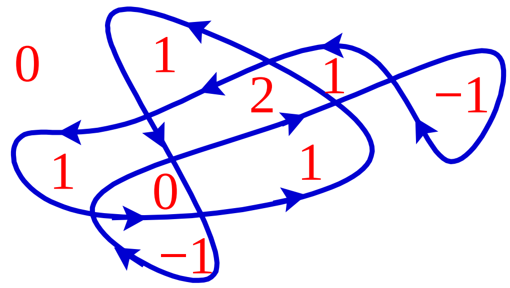

The harmonic extension for a function of faces.

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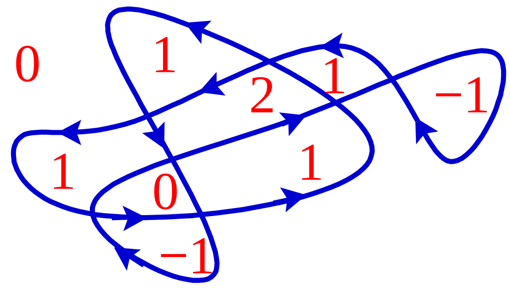

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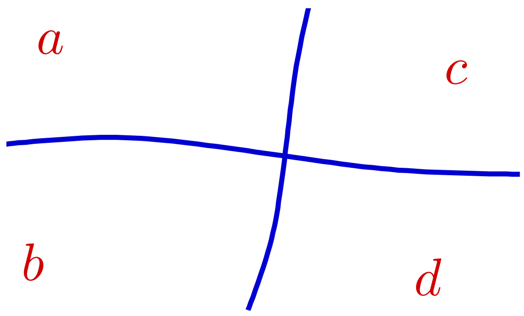

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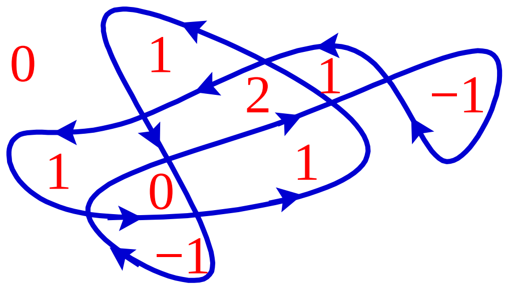

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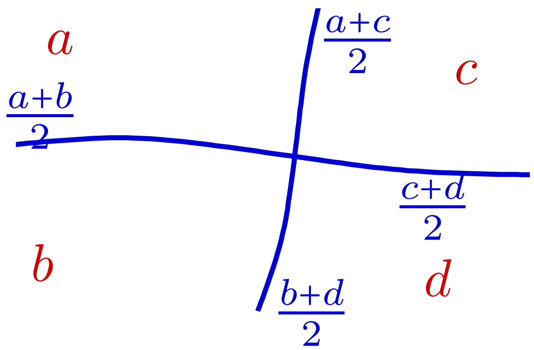

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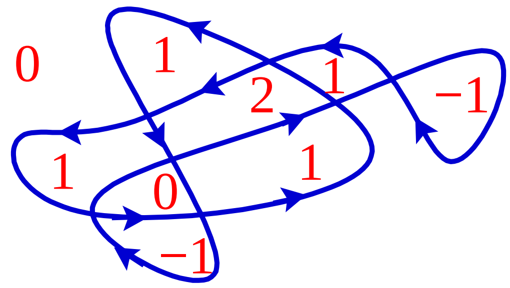

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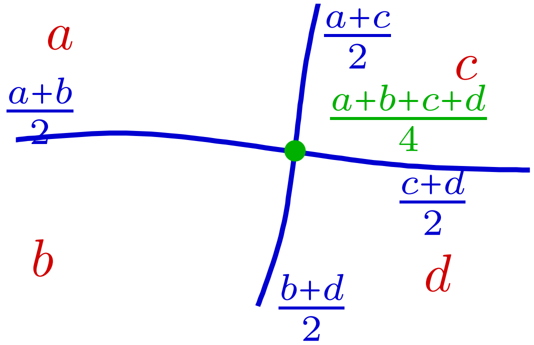

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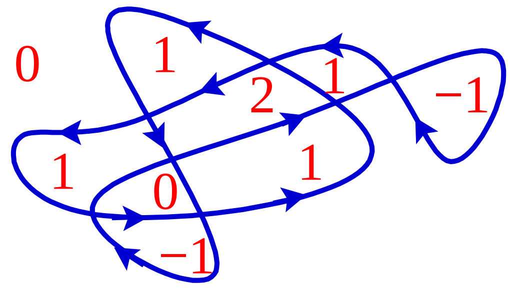

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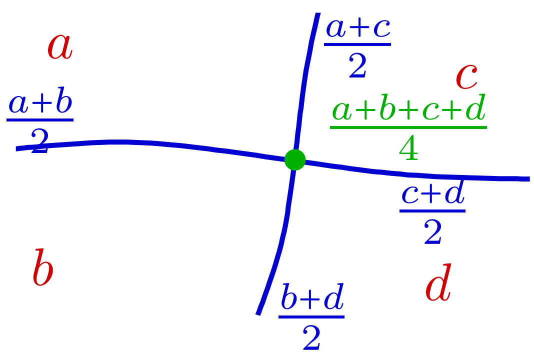

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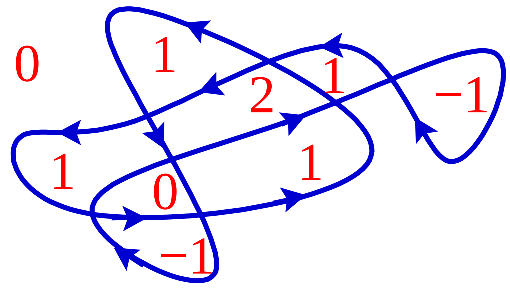

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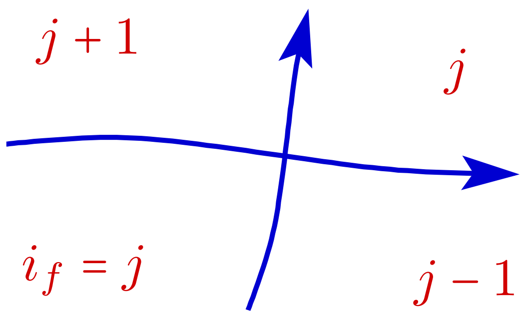

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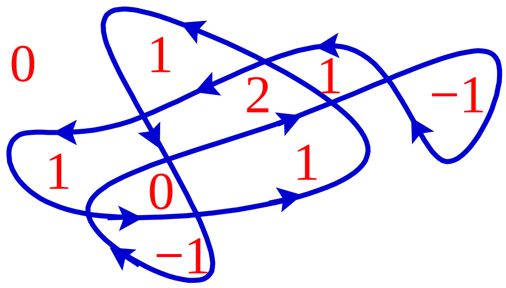

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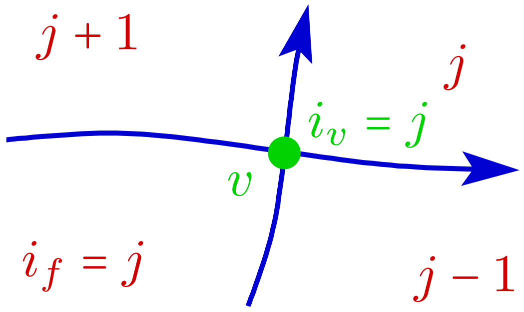

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Extra splitting of $e^{\pm}$, $i^{\pm}$, $J^{\pm}$ and n.

Planar circles

The space of circles on $\mathbb{R}^2$

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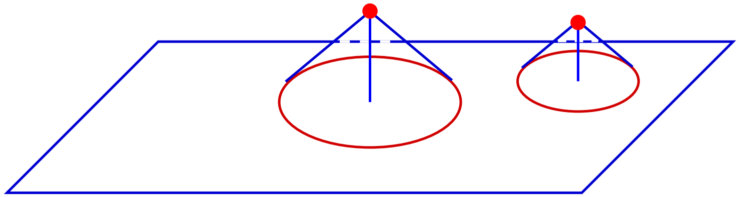

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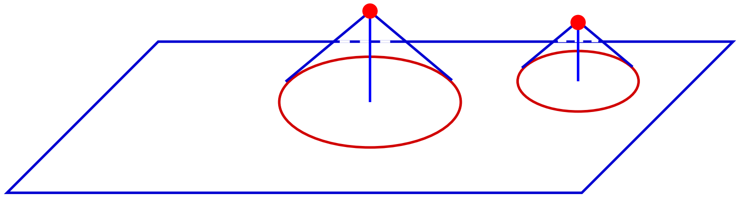
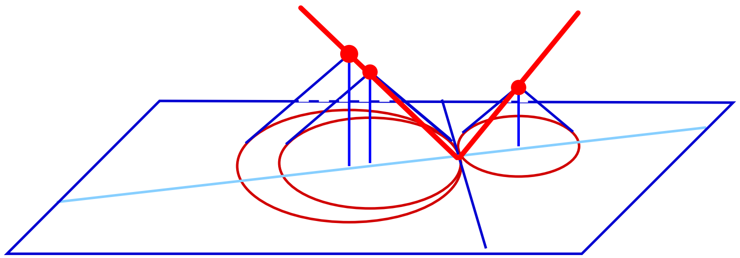


Circles tangent to a fixed line at a fixed point form two rays:



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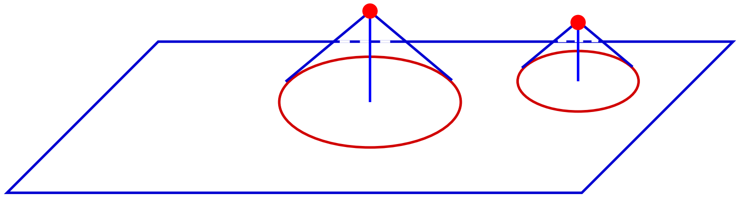

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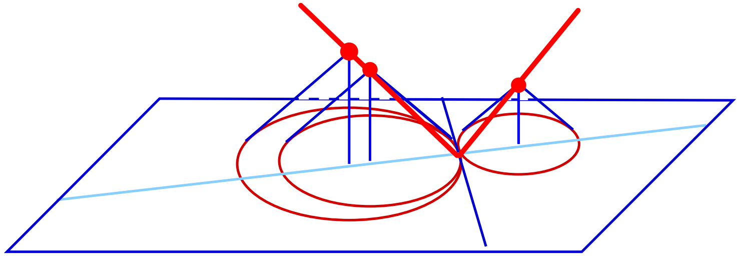
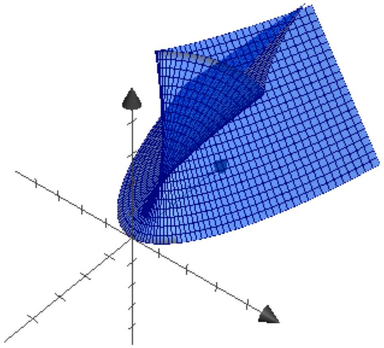


Circles tangent to a curve form a surface:



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Resolution of its multi-singularities

$$S = \{(c, p) \mid p \in S^1, c \text{ is tangent to } \gamma \text{ at } \gamma(p)\}$$

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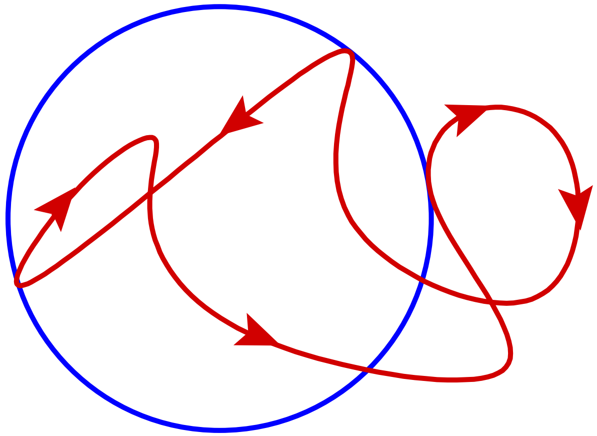

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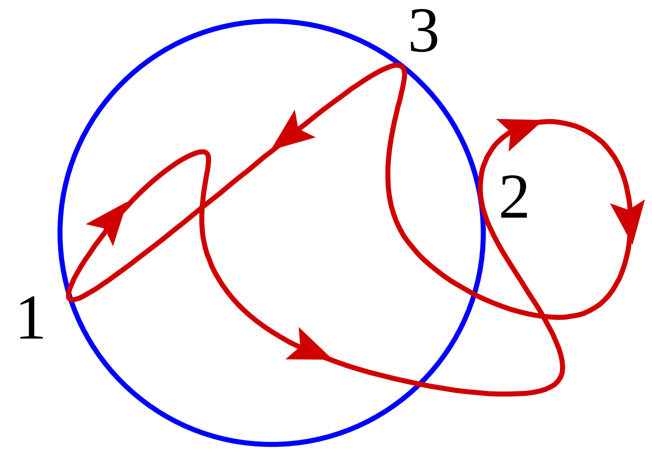

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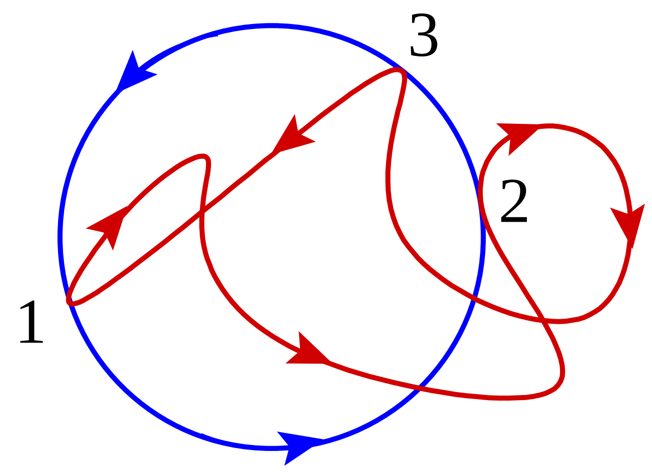

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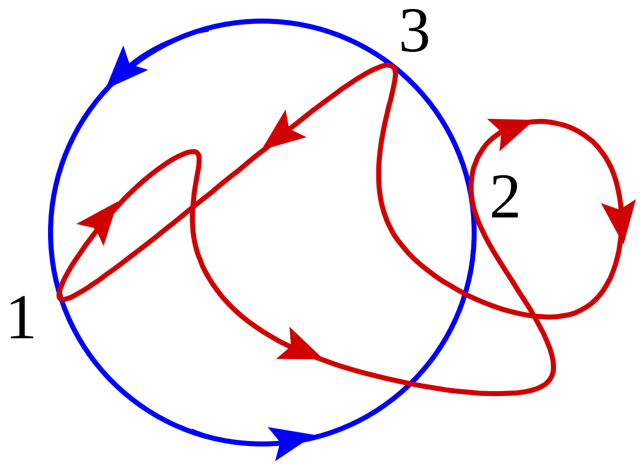

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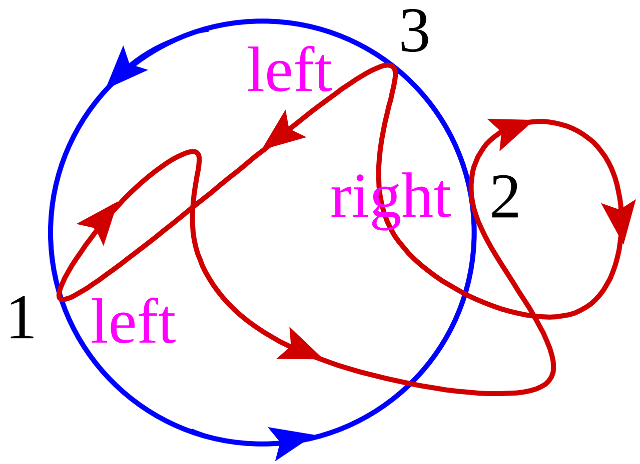

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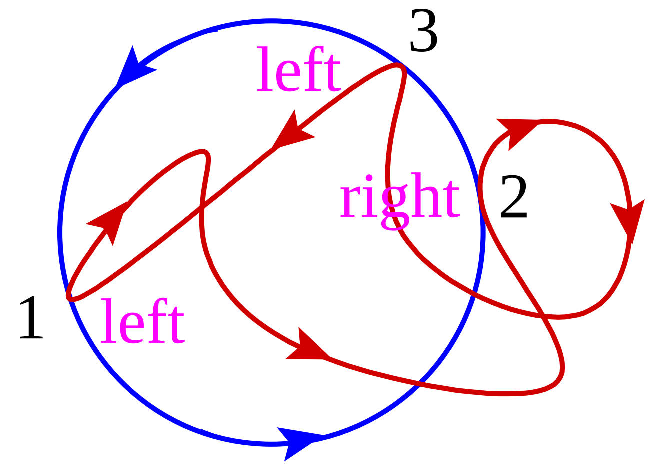

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The sign $\sigma(C)$ of the circle C is negative if the curve is
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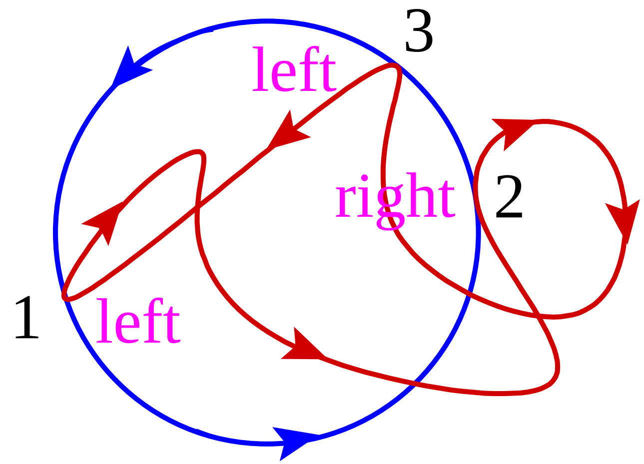

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On the picture, $\sigma = -1$.

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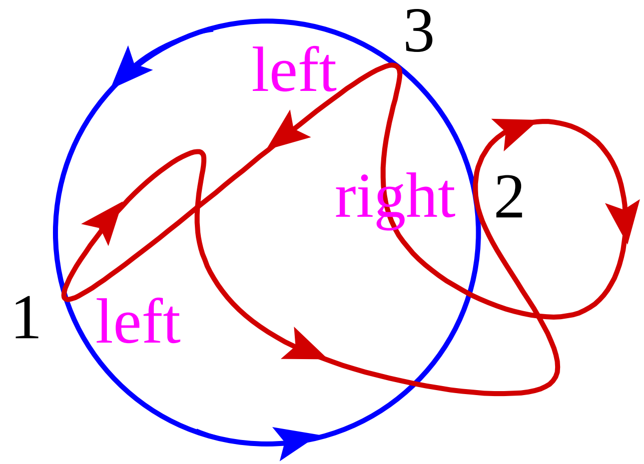

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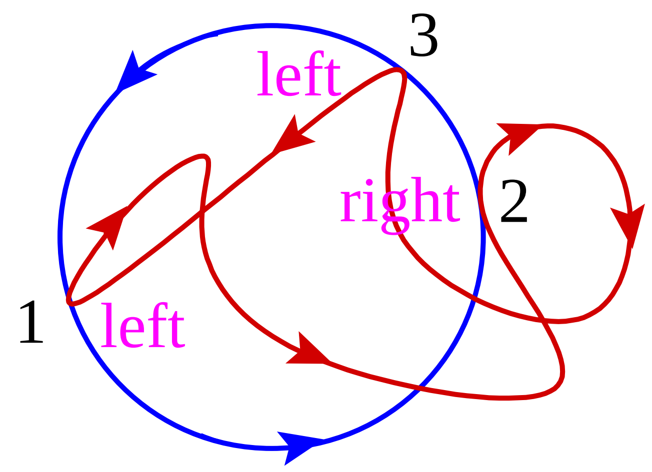

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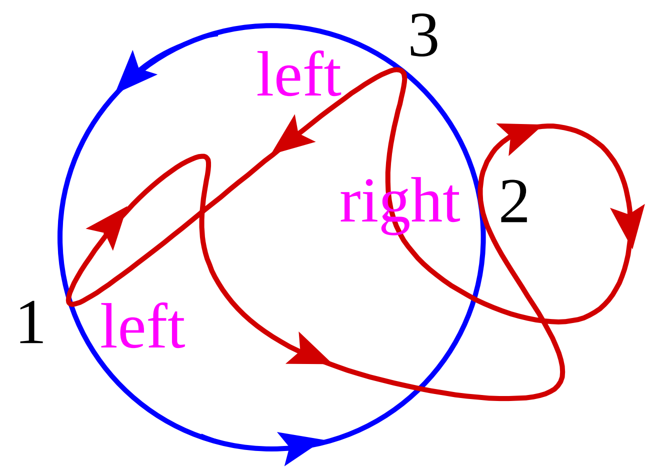

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Denote by T^i the set of tritangent circles with coherency i and put

$$t^i = \sum_{C \in T^i} \sigma(C).$$

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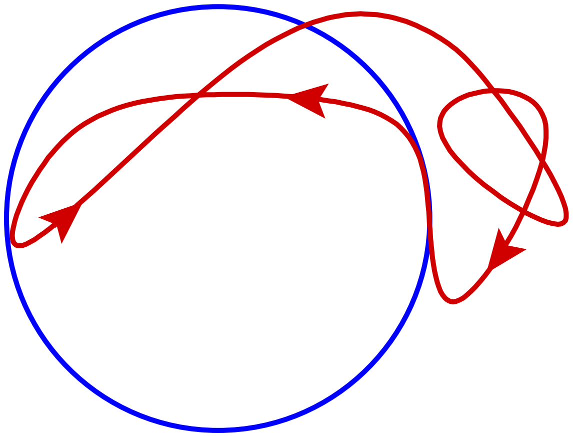

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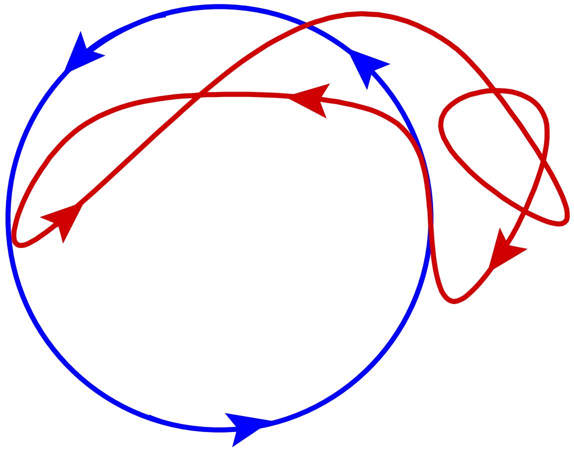

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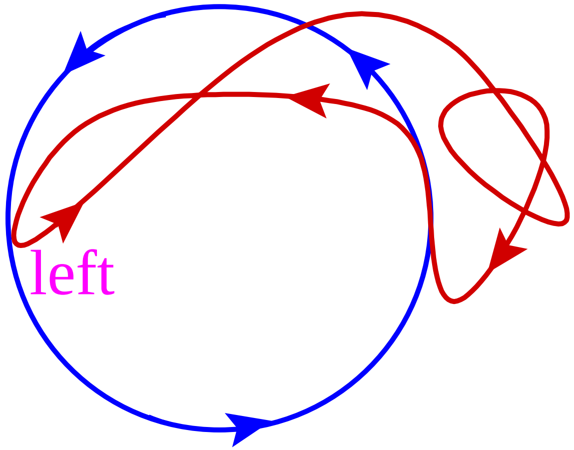

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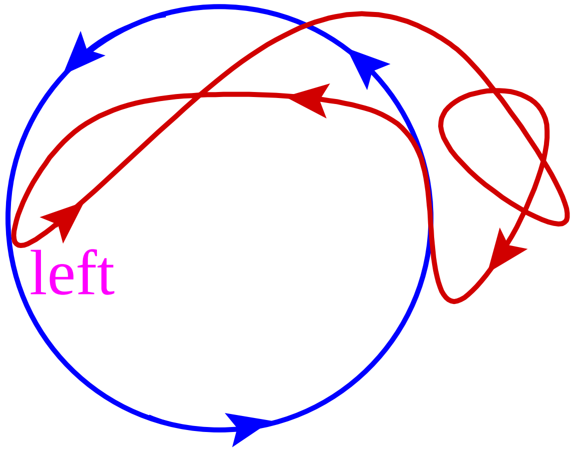

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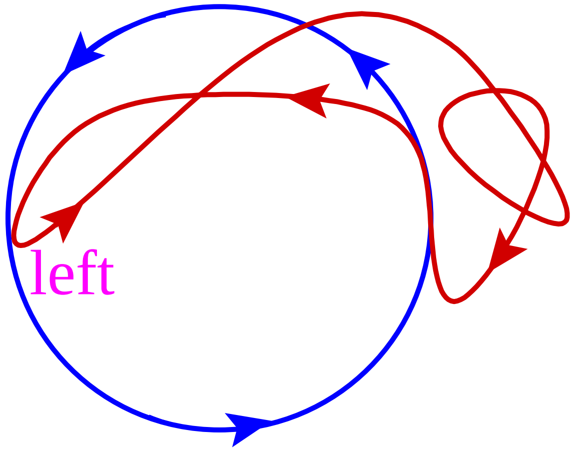

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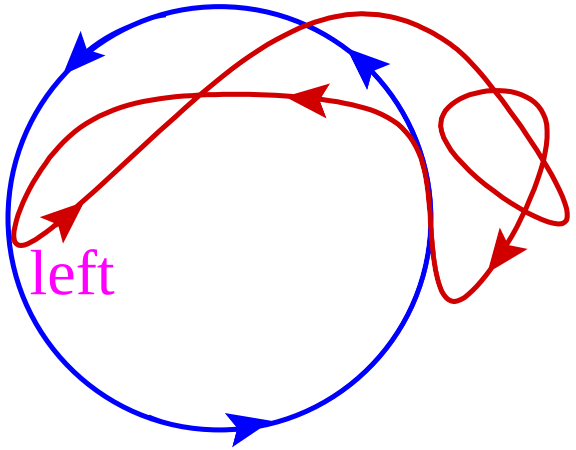

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Let $s^\pm = \sum_{C \in S^\pm} \sigma(C)$.

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Theorem (Yu. Sobolev). The numbers t^0, t^1, $\tau^2 = t^2 + \frac{s^-}{2}$ and $\tau^3 = t^3 + \frac{s^+}{2}$ are diffeomorphism invariants of γ. They change under the moves (perestroikas) of C as follows:

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	$\Delta(t^0)$	$\Delta(t^1)$	$\Delta(\tau^2)$	$\Delta(\tau^3)$
Triple point proper strong	-1	-3	3	1
Triple point reflected strong	1	3	-3	-1
Triple point proper weak	1	-1	1	-1
Triple point reflected weak	-1	1	-1	1
Direct self-tangency	$2\,\mathrm{ind}$	$-2\,\mathrm{ind}$	$2\,\mathrm{ind}$	$-2\,\mathrm{ind}$
Indirect left self-tangency	0	$4\,\mathrm{ind}-4$	$-4\,\mathrm{ind}+4$	0
Indirect right self-tangency	0	$4\,\mathrm{ind}+4$	$-4\,\mathrm{ind}-4$	0

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$t^0 = -\tau^3 = -\frac{1}{3}F + \frac{2}{3}E - V$ and $t^1 = -\tau^2 = F - \frac{2}{3}E + \frac{1}{3}V$