Twisted acyclicity of circle and link signatures

Oleg Viro

May 4, 2008

Twisted homology

Homology with coefficients in local system

Twisted homology

Homology with coefficients in local system,
a $\mathbb{C}$-bundle with a fixed flat connection,

Twisted homology

Homology with coefficients in local system,
a $\mathbb{C}$-bundle with a fixed flat connection,
that is an operation of parallel transport.

Twisted homology

Homology with coefficients in local system,
a $\mathbb{C}$-bundle with a fixed flat connection,
that is an operation of parallel transport.
It is defined by the monodromy representation $\pi_1(X) \to \mathbb{C}^\times$.

Twisted homology

Homology with coefficients in local system.

Simplicial model: chains are $\sum_\sigma a_\sigma \sigma$

Twisted homology

Homology with coefficients in local system.

Simplicial model: chains are $\sum_\sigma a_\sigma \sigma$, where σ are simplices and a_σ is a flat section over σ .

Twisted homology

Homology with coefficients in local system.

Simplicial model: chains are $\sum_\sigma a_\sigma \sigma$,
where σ are simplices and a_σ is a flat section over σ.

Differential involves restrictions of the sections to the faces.

Twisted homology

Homology with coefficients in local system.

Cellular model: chains are $\sum_\sigma a_\sigma \sigma$

Twisted homology

Homology with coefficients in local system.

Cellular model: chains are $\sum_\sigma a_\sigma \sigma$,
σ are oriented cells and a_σ is a flat section over σ .

Twisted homology

Homology with coefficients in local system.

Cellular model: chains are $\sum_\sigma a_\sigma \sigma$,
σ are oriented cells and a_σ is a flat section over σ .

Differential involves restrictions of the sections to the faces.

Twisted homology

Homology with coefficients in local system.

Singular model: chains are $\sum_\sigma a_\sigma \sigma$

Twisted homology

Homology with coefficients in local system.

Singular model: chains are $\sum_\sigma a_\sigma \sigma$,
σ are singular simplices and
a_σ is a flat section of the pull back of the coefficient bundle.

Twisted homology

Homology with coefficients in local system.

Singular model: chains are $\sum_\sigma a_\sigma \sigma$,
σ are singular simplices and
a_σ is a flat section of the pull back of the coefficient bundle.
Differential involves pull backs of the sections to the faces.

Twisted homology

Homology with coefficients in local system.

Theory is parallel to the untwisted homology theory.

Twisted homology

Homology with coefficients in local system.

Theory is parallel to the untwisted homology theory, but H_0 may be **trivial**.

Twisted homology

Homology with coefficients in local system.

H_0 may be **trivial**.

Example. $X = S^1$, with **non-trivial** monodromy $\pi_1(X) = \mathbb{Z} \to \mathbb{C}^\times$, say $\mu : 1 \mapsto a \neq 1$.

Twisted homology

Homology with coefficients in local system.

H_0 may be **trivial**.

Example. $X = S^1$, with **non-trivial** monodromy $\pi_1(X) = \mathbb{Z} \to \mathbb{C}^\times$, say $\mu : 1 \mapsto a \neq 1$. Then $\partial\sigma_1 = (a-1)\sigma_0 \neq 0$

Twisted homology

Homology with coefficients in local system.

H_0 may be **trivial**.

Example. $X = S^1$, with **non-trivial** monodromy $\pi_1(X) = \mathbb{Z} \to \mathbb{C}^\times$, say $\mu : 1 \mapsto a \neq 1$. Then $\partial\sigma_1 = (a-1)\sigma_0 \neq 0$, and $\mathbf{H_1}(\mathbf{X}; \mathbb{C}_\mu) = \mathbf{H_0}(\mathbf{X}; \mathbb{C}_\mu) = \mathbf{0}$.

Twisted homology

Homology with coefficients in local system.

H_0 may be **trivial**.

Example. $X = S^1$, with **non-trivial** monodromy $\pi_1(X) = \mathbb{Z} \to \mathbb{C}^\times$, say $\mu : 1 \mapsto a \neq 1$. Then $\partial\sigma_1 = (a-1)\sigma_0 \neq 0$, and $\mathbf{H_1(X; \mathbb{C}_\mu) = H_0(X; \mathbb{C}_\mu) = 0}$.

Generalization. $X = S^1 \times Y$, $\pi_1(X) = \mathbb{Z} \times \pi_1(Y)$.

Twisted homology

Homology with coefficients in local system.

H_0 may be **trivial**.

Example. $X = S^1$, with **non-trivial** monodromy $\pi_1(X) = \mathbb{Z} \to \mathbb{C}^\times$, say $\mu : 1 \mapsto a \neq 1$. Then $\partial\sigma_1 = (a-1)\sigma_0 \neq 0$, and $\mathbf{H_1}(\mathbf{X}; \mathbb{C}_\mu) = \mathbf{H_0}(\mathbf{X}; \mathbb{C}_\mu) = \mathbf{0}$.

Generalization. $X = S^1 \times Y$, $\pi_1(X) = \mathbb{Z} \times \pi_1(Y)$.
The monodromy is $\varphi \times \psi : \mathbb{Z} \times \pi_1(Y) \to \mathbb{C}^\times$.

Twisted homology

Homology with coefficients in local system.

H_0 may be **trivial**.

Example. $X = S^1$, with **non-trivial** monodromy $\pi_1(X) = \mathbb{Z} \to \mathbb{C}^\times$, say $\mu : 1 \mapsto a \neq 1$. Then $\partial\sigma_1 = (a-1)\sigma_0 \neq 0$, and $\mathbf{H_1}(\mathbf{X};\mathbb{C}_\mu) = \mathbf{H_0}(\mathbf{X};\mathbb{C}_\mu) = \mathbf{0}$.

Generalization. $X = S^1 \times Y$, $\pi_1(X) = \mathbb{Z} \times \pi_1(Y)$.
The monodromy is $\varphi \times \psi : \mathbb{Z} \times \pi_1(Y) \to \mathbb{C}^\times$.
Then $C_*(X;\mathbb{C}_{\varphi\times\psi}) = C_*(S^1;\mathbb{C}_\varphi) \otimes C_*(Y;\mathbb{C}_\psi)$ and
$H_*(X;\mathbb{C}_{\varphi\times\psi}) = H_*(S^1;\mathbb{C}_\phi) \otimes H_*(Y;\mathbb{C}_\psi)$

Twisted homology

Homology with coefficients in local system.

H_0 may be **trivial**.

Example. $X = S^1$, with **non-trivial** monodromy $\pi_1(X) = \mathbb{Z} \to \mathbb{C}^\times$, say $\mu : 1 \mapsto a \neq 1$. Then $\partial\sigma_1 = (a-1)\sigma_0 \neq 0$, and $\mathbf{H_1}(\mathbf{X};\mathbb{C}_\mu) = \mathbf{H_0}(\mathbf{X};\mathbb{C}_\mu) = \mathbf{0}$.

Generalization. $X = S^1 \times Y$, $\pi_1(X) = \mathbb{Z} \times \pi_1(Y)$. The monodromy is $\varphi \times \psi : \mathbb{Z} \times \pi_1(Y) \to \mathbb{C}^\times$. Then $C_*(X;\mathbb{C}_{\varphi\times\psi}) = C_*(S^1;\mathbb{C}_\varphi) \otimes C_*(Y;\mathbb{C}_\psi)$ and

$$H_*(X;\mathbb{C}_{\varphi\times\psi}) = \qquad \mathbf{0} \qquad \otimes H_*(Y;\mathbb{C}_\psi)$$

Twisted homology

Homology with coefficients in local system.

H_0 may be **trivial**.

Example. $X = S^1$, with **non-trivial** monodromy $\pi_1(X) = \mathbb{Z} \to \mathbb{C}^\times$, say $\mu : 1 \mapsto a \neq 1$. Then $\partial\sigma_1 = (a-1)\sigma_0 \neq 0$, and $\mathbf{H_1}(\mathbf{X}; \mathbb{C}_\mu) = \mathbf{H_0}(\mathbf{X}; \mathbb{C}_\mu) = \mathbf{0}$.

Generalization. $X = S^1 \times Y$, $\pi_1(X) = \mathbb{Z} \times \pi_1(Y)$.
The monodromy is $\varphi \times \psi : \mathbb{Z} \times \pi_1(Y) \to \mathbb{C}^\times$.
Then $C_*(X; \mathbb{C}_{\varphi\times\psi}) = C_*(S^1; \mathbb{C}_\varphi) \otimes C_*(Y; \mathbb{C}_\psi)$ and
$H_*(X; \mathbb{C}_{\varphi\times\psi}) = \mathbf{0} \otimes H_*(Y; \mathbb{C}_\psi) = \mathbf{0}$.

Twisted homology

Homology with coefficients in local system.

H_0 may be **trivial**.

Example. $X = S^1$, with **non-trivial** monodromy $\pi_1(X) = \mathbb{Z} \to \mathbb{C}^\times$, say $\mu : 1 \mapsto a \neq 1$. Then $\partial\sigma_1 = (a-1)\sigma_0 \neq 0$, and $\mathbf{H_1(X;\mathbb{C}_\mu) = H_0(X;\mathbb{C}_\mu) = 0}$.

Generalization. $X = S^1 \times Y$, $\pi_1(X) = \mathbb{Z} \times \pi_1(Y)$. The monodromy is $\varphi \times \psi : \mathbb{Z} \times \pi_1(Y) \to \mathbb{C}^\times$. Then $C_*(X;\mathbb{C}_{\varphi\times\psi}) = C_*(S^1;\mathbb{C}_\varphi) \otimes C_*(Y;\mathbb{C}_\psi)$ and $H_*(X;\mathbb{C}_{\varphi\times\psi}) = \quad \mathbf{0} \quad \otimes H_*(Y;\mathbb{C}_\psi) = \mathbf{0}$.

Furthermore, the same holds true for any locally trivial fibration with fiber S^1 and non-trivial monodromy along the fiber.

Twisted homology

Homology with coefficients in local system.

H_0 may be **trivial**.

Example. $X = S^1$, with **non-trivial** monodromy $\pi_1(X) = \mathbb{Z} \to \mathbb{C}^\times$, say $\mu : 1 \mapsto a \neq 1$. Then $\partial\sigma_1 = (a-1)\sigma_0 \neq 0$, and $\mathbf{H_1}(\mathbf{X}; \mathbb{C}_\mu) = \mathbf{H_0}(\mathbf{X}; \mathbb{C}_\mu) = \mathbf{0}$.

Generalization. $X = S^1 \times Y$, $\pi_1(X) = \mathbb{Z} \times \pi_1(Y)$.
The monodromy is $\varphi \times \psi : \mathbb{Z} \times \pi_1(Y) \to \mathbb{C}^\times$.
Then $C_*(X; \mathbb{C}_{\varphi\times\psi}) = C_*(S^1; \mathbb{C}_\varphi) \otimes C_*(Y; \mathbb{C}_\psi)$ and
$H_*(X; \mathbb{C}_{\varphi\times\psi}) = \quad \mathbf{0} \quad \otimes H_*(Y; \mathbb{C}_\psi) \; = \; \mathbf{0}$.

Furthermore, the same holds true for any locally trivial fibration with fiber S^1 and non-trivial monodromy along the fiber.
Pieces of a space of this kind are

invisible for twisted homology.

Duality

Let X be a connected oriented compact manifold of dim n.

Duality

Let X be a connected oriented compact manifold of dim n.
$H_n(X, \partial X) = \mathbb{Z}$, $H_n(X, \partial X; \mathbb{C}) = \mathbb{C}$,
an orientation of X = a generator of $H_n(X, \partial X)$.

Duality

Let X be a connected oriented compact manifold of dim n.

$H_n(X, \partial X) = \mathbb{Z}$, $H_n(X, \partial X; \mathbb{C}) = \mathbb{C}$,

an orientation of X = a generator of $H_n(X, \partial X)$.

Poincaré duality isomorphisms:

$H^p(X; \mathbb{C}_\mu) \to H_{n-p}(X, \partial X; \mathbb{C}_\mu)$ and

$H^p(X, \partial X; \mathbb{C}_\mu) \to H_{n-p}(X; \mathbb{C}_\mu)$.

Duality

Let X be a connected oriented compact manifold of dim n.

$H_n(X, \partial X) = \mathbb{Z}$, $H_n(X, \partial X; \mathbb{C}) = \mathbb{C}$,

an orientation of X = a generator of $H_n(X, \partial X)$.

Poincaré duality isomorphisms:

$H^p(X; \mathbb{C}_\mu) \to H_{n-p}(X, \partial X; \mathbb{C}_\mu)$ and

$H^p(X, \partial X; \mathbb{C}_\mu) \to H_{n-p}(X; \mathbb{C}_\mu)$.

Pairings of local coefficient systems: $\mathbb{C}_\mu \otimes \mathbb{C}_{\mu^{-1}} = \mathbb{C}$.

Duality

Let X be a connected oriented compact manifold of dim n.
$H_n(X, \partial X) = \mathbb{Z}$, $H_n(X, \partial X; \mathbb{C}) = \mathbb{C}$,
an orientation of X = a generator of $H_n(X, \partial X)$.
Poincaré duality isomorphisms:
$H^p(X; \mathbb{C}_\mu) \to H_{n-p}(X, \partial X; \mathbb{C}_\mu)$ and
$H^p(X, \partial X; \mathbb{C}_\mu) \to H_{n-p}(X; \mathbb{C}_\mu)$.
Pairings of local coefficient systems: $\mathbb{C}_\mu \otimes \mathbb{C}_{\mu^{-1}} = \mathbb{C}$
induces a non-singular bilinear intersection pairing
$H_p(X, \partial X; \mathbb{C}_\mu) \otimes H_{n-p}(X; \mathbb{C}_{\mu^{-1}}) \to \mathbb{C}$.

Unitary local coefficients

Unitary local coefficients

Representation $\mu : \pi_1(X) \longrightarrow \mathbb{C}^\times$ is unitary if $\mu^{-1} = \bar{\mu}$.

Unitary local coefficients

Representation $\mu : \pi_1(X) \longrightarrow \mathbb{C}^\times$ is unitary if $\mu^{-1} = \bar{\mu}$.

pointwise: $(\mu(\alpha))^{-1} = \overline{\mu(\alpha)}$ for any $\alpha \in \pi_1(X)$.

Unitary local coefficients

Representation $\mu : \pi_1(X) \longrightarrow \mathbb{C}^\times$ is unitary if $\mu^{-1} = \bar{\mu}$.

If μ is unitary, then the conjugation induces a semilinear bijection $H_q(X; \mathbb{C}_\mu) \to H_q(X; \mathbb{C}_{\bar{\mu}}) = H_q(X; \mathbb{C}_{\mu^{-1}})$.

Unitary local coefficients

Representation $\mu : \pi_1(X) \to \mathbb{C}^\times$ is unitary if $\mu^{-1} = \bar{\mu}$.

If μ is unitary, then the conjugation induces a semilinear bijection $H_q(X; \mathbb{C}_\mu) \to H_q(X; \mathbb{C}_{\bar{\mu}}) = H_q(X; \mathbb{C}_{\mu^{-1}})$.

In the case of oriented compact n-dimensional manifold, it turns a non-singular bilinear intersection pairing

$H_p(X, \partial X; \mathbb{C}_\mu) \otimes H_{n-p}(X; \mathbb{C}_{\mu^{-1}}) \to \mathbb{C}$

into a non-singular sesqui-linear intersection pairing

$H_p(X, \partial X; \mathbb{C}_\mu) \otimes H_{n-p}(X; \mathbb{C}_\mu) \to \mathbb{C}$.

Unitary local coefficients

Representation $\mu : \pi_1(X) \to \mathbb{C}^\times$ is unitary if $\mu^{-1} = \bar{\mu}$.
If μ is unitary, then the conjugation induces a semilinear bijection
$H_q(X;\mathbb{C}_\mu) \to H_q(X;\mathbb{C}_{\bar{\mu}}) = H_q(X;\mathbb{C}_{\mu^{-1}})$.
In the case of oriented compact n-dimensional manifold, it turns a non-singular bilinear intersection pairing
$H_p(X,\partial X;\mathbb{C}_\mu) \otimes H_{n-p}(X;\mathbb{C}_{\mu^{-1}}) \to \mathbb{C}$
into a non-singular sesqui-linear intersection pairing
$H_p(X,\partial X;\mathbb{C}_\mu) \otimes H_{n-p}(X;\mathbb{C}_\mu) \to \mathbb{C}$,
composed with relativization, it gives
$H_p(X;\mathbb{C}_\mu) \otimes H_{n-p}(X;\mathbb{C}_\mu) \to \mathbb{C}$.

Unitary local coefficients

Representation $\mu : \pi_1(X) \to \mathbb{C}^\times$ is unitary if $\mu^{-1} = \bar{\mu}$.
If μ is unitary, then the conjugation induces a semilinear bijection $H_q(X; \mathbb{C}_\mu) \to H_q(X; \mathbb{C}_{\bar{\mu}}) = H_q(X; \mathbb{C}_{\mu^{-1}})$.
In the case of oriented compact n-dimensional manifold, it turns a non-singular bilinear intersection pairing $H_p(X, \partial X; \mathbb{C}_\mu) \otimes H_{n-p}(X; \mathbb{C}_{\mu^{-1}}) \to \mathbb{C}$
into a non-singular sesqui-linear intersection pairing $H_p(X, \partial X; \mathbb{C}_\mu) \otimes H_{n-p}(X; \mathbb{C}_\mu) \to \mathbb{C}$,
composed with relativization, it gives $H_p(X; \mathbb{C}_\mu) \otimes H_{n-p}(X; \mathbb{C}_\mu) \to \mathbb{C}$.
In the middle dimension this is a Hermitian or skew-Hermitian form.

Unitary local coefficients

Representation $\mu : \pi_1(X) \longrightarrow \mathbb{C}^\times$ is unitary if $\mu^{-1} = \bar{\mu}$.

If μ is unitary, then the conjugation induces a semilinear bijection $H_q(X; \mathbb{C}_\mu) \longrightarrow H_q(X; \mathbb{C}_{\bar{\mu}}) = H_q(X; \mathbb{C}_{\mu^{-1}})$.

In the case of oriented compact n-dimensional manifold, it turns a non-singular bilinear intersection pairing

$$H_p(X, \partial X; \mathbb{C}_\mu) \otimes H_{n-p}(X; \mathbb{C}_{\mu^{-1}}) \longrightarrow \mathbb{C}$$

into a non-singular sesqui-linear intersection pairing

$$H_p(X, \partial X; \mathbb{C}_\mu) \otimes H_{n-p}(X; \mathbb{C}_\mu) \longrightarrow \mathbb{C},$$

composed with relativization, it gives

$$H_p(X; \mathbb{C}_\mu) \otimes H_{n-p}(X; \mathbb{C}_\mu) \longrightarrow \mathbb{C}.$$

In the middle dimension this is a Hermitian or skew-Hermitian form.

If $\partial X = \varnothing$, then the intersection pairing is non-singular.

Unitary local coefficients

Representation $\mu : \pi_1(X) \to \mathbb{C}^\times$ is unitary if $\mu^{-1} = \bar{\mu}$.

If μ is unitary, then the conjugation induces a semilinear bijection $H_q(X; \mathbb{C}_\mu) \to H_q(X; \mathbb{C}_{\bar{\mu}}) = H_q(X; \mathbb{C}_{\mu^{-1}})$.

In the case of oriented compact n-dimensional manifold, it turns a non-singular bilinear intersection pairing

$$H_p(X, \partial X; \mathbb{C}_\mu) \otimes H_{n-p}(X; \mathbb{C}_{\mu^{-1}}) \to \mathbb{C}$$

into a non-singular sesqui-linear intersection pairing

$$H_p(X, \partial X; \mathbb{C}_\mu) \otimes H_{n-p}(X; \mathbb{C}_\mu) \to \mathbb{C},$$

composed with relativization, it gives

$$H_p(X; \mathbb{C}_\mu) \otimes H_{n-p}(X; \mathbb{C}_\mu) \to \mathbb{C}.$$

In the middle dimension this is a Hermitian or skew-Hermitian form.

If $\partial X = \varnothing$, or ∂X is fibered with fibre S^1, then the intersection pairing is non-singular.

Signatures

Let M be a compact oriented $2n$-dimensional manifold

Signatures

Let M be a compact oriented $2n$-dimensional manifold, $L_1, \ldots, L_k$ its oriented compact $(2n-2)$-dimensional submanifolds transversal to each other with $\partial L_i = L_i \cap \partial M$, let $L = \cup_i L_i$.

Signatures

Let M be a compact oriented $2n$-dimensional manifold, $L_1, \ldots, L_k$ its oriented compact $(2n-2)$-dimensional submanifolds transversal to each other with $\partial L_i = L_i \cap \partial M$, let $L = \cup_i L_i$.

Let $\mu \in \mathrm{Hom}(H_1(M \smallsetminus L), \mathbb{C}^\times)$, and $\mathbb{C}_\mu$ be the corresponding local coefficient system on $M \smallsetminus L$.

Signatures

Let M be a compact oriented $2n$-dimensional manifold, $L_1, \ldots, L_k$ its oriented compact $(2n-2)$-dimensional submanifolds transversal to each other with $\partial L_i = L_i \cap \partial M$, let $L = \cup_i L_i$.
Let $\mu \in \mathrm{Hom}(H_1(M \smallsetminus L), \mathbb{C}^\times)$, and $\mathbb{C}_\mu$ be the corresponding local coefficient system on $M \smallsetminus L$.

If n is even, then denote by $\sigma_\mu(M \smallsetminus L)$ the signature of the Hermitian intersection form in $H_n(M \smallsetminus L; \mathbb{C}_\mu)$.

Signatures

Let M be a compact oriented $2n$-dimensional manifold, $L_1, \dots, L_k$ its oriented compact $(2n-2)$-dimensional submanifolds transversal to each other with $\partial L_i = L_i \cap \partial M$, let $L = \cup_i L_i$.
Let $\mu \in \mathrm{Hom}(H_1(M \smallsetminus L), \mathbb{C}^\times)$, and $\mathbb{C}_\mu$ be the corresponding local coefficient system on $M \smallsetminus L$.

If n is even, then denote by $\sigma_\mu(M \smallsetminus L)$ the signature of the Hermitian intersection form in $H_n(M \smallsetminus L; \mathbb{C}_\mu)$.

If n is odd, then denote by $\sigma_\mu(M \smallsetminus L)$ the signature of the Hermitian form obtained from the skew-Hermitian intersection form in $H_n(M \smallsetminus L; \mathbb{C}_\mu)$ multiplied by $\sqrt{-1}$.

Signatures

Let M be a compact oriented $2n$-dimensional manifold, $L_1, \dots, L_k$ its oriented compact $(2n-2)$-dimensional submanifolds transversal to each other with $\partial L_i = L_i \cap \partial M$, let $L = \cup_i L_i$.
Let $\mu \in \mathrm{Hom}(H_1(M \smallsetminus L), \mathbb{C}^\times)$, and $\mathbb{C}_\mu$ be the corresponding local coefficient system on $M \smallsetminus L$.

Properties of signatures

1. If W is an oriented compact manifold, $M = \partial W$,

Signatures

Let M be a compact oriented $2n$-dimensional manifold, $L_1, \dots, L_k$ its oriented compact $(2n-2)$-dimensional submanifolds transversal to each other with $\partial L_i = L_i \cap \partial M$, let $L = \cup_i L_i$.
Let $\mu \in \mathrm{Hom}(H_1(M \setminus L), \mathbb{C}^\times)$, and $\mathbb{C}_\mu$ be the corresponding local coefficient system on $M \setminus L$.

Properties of signatures

1. If W is an oriented compact manifold, $M = \partial W$,

In particular, $\partial M = \varnothing$ and $\partial L_i = \varnothing$

Signatures

Let M be a compact oriented $2n$-dimensional manifold, $L_1, \dots, L_k$ its oriented compact $(2n-2)$-dimensional submanifolds transversal to each other with $\partial L_i = L_i \cap \partial M$, let $L = \cup_i L_i$.
Let $\mu \in \mathrm{Hom}(H_1(M \smallsetminus L), \mathbb{C}^\times)$, and $\mathbb{C}_\mu$ be the corresponding local coefficient system on $M \smallsetminus L$.

Properties of signatures

1. If W is an oriented compact manifold, $M = \partial W$, and $F_i \subset W$ are compact oriented transversal to each other, $L_i = \partial F_i$

Signatures

Let M be a compact oriented $2n$-dimensional manifold, $L_1, \ldots, L_k$ its oriented compact $(2n-2)$-dimensional submanifolds transversal to each other with $\partial L_i = L_i \cap \partial M$, let $L = \cup_i L_i$.
Let $\mu \in \mathrm{Hom}(H_1(M \smallsetminus L), \mathbb{C}^\times)$, and $\mathbb{C}_\mu$ be the corresponding local coefficient system on $M \smallsetminus L$.

Properties of signatures

1. If W is an oriented compact manifold, $M = \partial W$, and $F_i \subset W$ are compact oriented transversal to each other, $L_i = \partial F_i$, then $\sigma_\mu(M \smallsetminus L) = 0$.

Signatures

2. Let M' be another compact oriented $2n$-dimensional manifold, $L'_1, \ldots, L'_k$ its oriented compact $(2n-2)$-dimensional submanifolds transversal to each other with $\partial L'_i = L'_i \cap \partial M'$, and $L' = \cup_i L'_i$.

Signatures

2. Let M' be another compact oriented $2n$-dimensional manifold, $L'_1, \dots, L'_k$ its oriented compact $(2n-2)$-dimensional submanifolds transversal to each other with $\partial L'_i = L'_i \cap \partial M'$, and $L' = \cup_i L'_i$. Let $M \cap M' = \partial M \cap \partial M'$ be a compact manifold of dimension $2n-1$ and the orientations induced on $M \cap M'$ from M and M' are opposite to each other.

Signatures

2. Let M' be another compact oriented $2n$ -dimensional manifold, $L'_1, \dots, L'_k$ its oriented compact $(2n-2)$ -dimensional submanifolds transversal to each other with $\partial L'_i = L'_i \cap \partial M'$, and $L' = \cup_i L'_i$. Let $M \cap M' = \partial M \cap \partial M'$ be a compact manifold of dimension $2n-1$ and the orientations induced on $M \cap M'$ from M and M' are opposite to each other.
Let $\mu' \in \mathrm{Hom}(H_1(M' \smallsetminus L'), \mathbb{C}^\times)$, and $\mathbb{C}_{\mu'}$ be the corresponding local coefficient system on $M' \smallsetminus L'$ and $\mathbb{C}_\mu|_{M \cap M'} = \mathbb{C}_{\mu'}|_{M \cap M'}$.

Signatures

2. Let M' be another compact oriented $2n$-dimensional manifold, $L'_1, \dots, L'_k$ its oriented compact $(2n-2)$-dimensional submanifolds transversal to each other with $\partial L'_i = L'_i \cap \partial M'$, and $L' = \cup_i L'_i$. Let $M \cap M' = \partial M \cap \partial M'$ be a compact manifold of dimension $2n-1$ and the orientations induced on $M \cap M'$ from M and M' are opposite to each other.
Let $\mu' \in \mathrm{Hom}(H_1(M' \smallsetminus L'), \mathbb{C}^\times)$, and $\mathbb{C}_{\mu'}$ be the corresponding local coefficient system on $M' \smallsetminus L'$ and $\mathbb{C}_\mu|_{M\cap M'} = \mathbb{C}_{\mu'}|_{M\cap M'}$. Assume that $\partial(M \cap M')$ is fibered with fibers circles on which μ is non-trivial.

Signatures

2. Let M' be another compact oriented $2n$-dimensional manifold, $L'_1, \dots, L'_k$ its oriented compact $(2n-2)$-dimensional submanifolds transversal to each other with $\partial L'_i = L'_i \cap \partial M'$, and $L' = \cup_i L'_i$. Let $M \cap M' = \partial M \cap \partial M'$ be a compact manifold of dimension $2n-1$ and the orientations induced on $M \cap M'$ from M and M' are opposite to each other.
Let $\mu' \in \operatorname{Hom}(H_1(M' \smallsetminus L'), \mathbb{C}^\times)$, and $\mathbb{C}_{\mu'}$ be the corresponding local coefficient system on $M' \smallsetminus L'$ and $\mathbb{C}_\mu|_{M\cap M'} = \mathbb{C}_{\mu'}|_{M\cap M'}$. Assume that $\partial(M \cap M')$ is fibered with fibers circles on which μ is non-trivial. Then $\sigma_{\mu\cup\mu'}((M \cup M') \smallsetminus (L \cup L')) = \sigma_\mu(M \smallsetminus L) + \sigma_{\mu'}(M' \smallsetminus L')$.

Signatures

Corollary. $\sigma_\mu(M \smallsetminus L)$ is invariant with respect to cobordisms of $(M; L_1, \ldots, L_k; \mu)$.

Link signatures

Let $L = L_1 \cup \cdots \cup L_m \subset S^3$ be a classical link.

Link signatures

Let $L = L_1 \cup \cdots \cup L_m \subset S^3$ be a classical link.
$\zeta_i \in \mathbb{C}$, $|\zeta_i| = 1$, $\zeta = (\zeta_1, \ldots, \zeta_m) \in (S^1)^m$ and
$\mu : \pi_1(S^3 \setminus L) \to \mathbb{C}^\times$ takes a meridian of L_i to ζ_i.

Link signatures

Let $L = L_1 \cup \cdots \cup L_m \subset S^3$ be a classical link.
$\zeta_i \in \mathbb{C}$, $|\zeta_i| = 1$, $\zeta = (\zeta_1, \dots, \zeta_m) \in (S^1)^m$ and
$\mu : \pi_1(S^3 \setminus L) \to \mathbb{C}^\times$ takes a meridian of L_i to ζ_i.
Let $F_i \subset D^4$ be smooth oriented surfaces transversal to each other with $\partial F_i = F_i \cap \partial D^4 = L_i$.

Link signatures

Let $L = L_1 \cup \cdots \cup L_m \subset S^3$ be a classical link.
$\zeta_i \in \mathbb{C}$, $|\zeta_i| = 1$, $\zeta = (\zeta_1, \ldots, \zeta_m) \in (S^1)^m$ and
$\mu : \pi_1(S^3 \smallsetminus L) \longrightarrow \mathbb{C}^\times$ takes a meridian of L_i to ζ_i.
Let $F_i \subset D^4$ be smooth oriented surfaces transversal to each other with $\partial F_i = F_i \cap \partial D^4 = L_i$. Extend μ to $D^4 \smallsetminus \cup_i F_i$.

Link signatures

Let $L = L_1 \cup \cdots \cup L_m \subset S^3$ be a classical link.
$\zeta_i \in \mathbb{C}$, $|\zeta_i| = 1$, $\zeta = (\zeta_1, \ldots, \zeta_m) \in (S^1)^m$ and
$\mu : \pi_1(S^3 \smallsetminus L) \longrightarrow \mathbb{C}^\times$ takes a meridian of L_i to ζ_i.
Let $F_i \subset D^4$ be smooth oriented surfaces transversal to each other with $\partial F_i = F_i \cap \partial D^4 = L_i$. Extend μ to $D^4 \smallsetminus \cup_i F_i$.
In $H_2(D^4 \smallsetminus \cup_i F_i; \mathbb{C}_\mu)$ there is a Hermitian intersection form.

Link signatures

In $H_2(D^4 \smallsetminus \cup_i F_i; \mathbb{C}_\mu)$ there is a Hermitian intersection form.

Theorem. *Its signature* $\sigma_\zeta(L)$ *does not depend on* $F_1, \ldots, F_m$.

Link signatures

In $H_2(D^4 \setminus \cup_i F_i; \mathbb{C}_\mu)$ there is a Hermitian intersection form.

Theorem. *Its signature* $\sigma_\zeta(L)$ *does not depend on* $F_1, \ldots, F_m$*.*

Proof. Any F_i' with $\partial F_i' = F_i' \cap \partial D^4 = l_i$ is cobordant to F_i.

Link signatures

In $H_2(D^4 \smallsetminus \cup_i F_i; \mathbb{C}_\mu)$ there is a Hermitian intersection form.

Theorem. *Its signature* $\sigma_\zeta(L)$ *does not depend on* $F_1, \dots, F_m$ *.*

Proof. Any F_i' with $\partial F_i' = F_i' \cap \partial D^4 = l_i$ is cobordant to F_i . The cobordisms $W_i \subset D^4 \times I$ can be made pairwise transversal.

Link signatures

In $H_2(D^4 \smallsetminus \cup_i F_i; \mathbb{C}_\mu)$ there is a Hermitian intersection form.

Theorem. *Its signature* $\sigma_\zeta(L)$ *does not depend on* $F_1, \dots, F_m$ *.*

Proof. Any F_i' with $\partial F_i' = F_i' \cap \partial D^4 = l_i$ is cobordant to F_i . The cobordisms $W_i \subset D^4 \times I$ can be made pairwise transversal. They define a cobordism $D^4 \times I \smallsetminus \cup_i N(W_i)$ between $D^4 \smallsetminus \cup_i N(F_i)$ and $D^4 \smallsetminus \cup_i N(F_i')$.

Link signatures

In $H_2(D^4 \smallsetminus \cup_i F_i; \mathbb{C}_\mu)$ there is a Hermitian intersection form.

Theorem. *Its signature* $\sigma_\zeta(L)$ *does not depend on* $F_1, \dots, F_m$ *.*

Proof. Any F_i' with $\partial F_i' = F_i' \cap \partial D^4 = l_i$ is cobordant to F_i . The cobordisms $W_i \subset D^4 \times I$ can be made pairwise transversal. They define a cobordism $D^4 \times I \smallsetminus \cup_i N(W_i)$ between $D^4 \smallsetminus \cup_i N(F_i)$ and $D^4 \smallsetminus \cup_i N(F_i')$. The boundary of the cobordism consists of $D^4 \smallsetminus \cup_i N(F_i)$, $D^4 \smallsetminus \cup_i N(F_i')$ and a homologically negligible part

Link signatures

In $H_2(D^4 \smallsetminus \cup_i F_i; \mathbb{C}_\mu)$ there is a Hermitian intersection form.

Theorem. *Its signature* $\sigma_\zeta(L)$ *does not depend on* $F_1, \dots, F_m$ *.*

Proof. Any F_i' with $\partial F_i' = F_i' \cap \partial D^4 = l_i$ is cobordant to F_i . The cobordisms $W_i \subset D^4 \times I$ can be made pairwise transversal. They define a cobordism $D^4 \times I \smallsetminus \cup_i N(W_i)$ between $D^4 \smallsetminus \cup_i N(F_i)$ and $D^4 \smallsetminus \cup_i N(F_i')$. The boundary of the cobordism consists of $D^4 \smallsetminus \cup_i N(F_i)$, $D^4 \smallsetminus \cup_i N(F_i')$ and a homologically negligible part $\partial(N(\cup_i W_i))$, the boundary of a regular neighborhood of the cobordism $\cup_i W_i$ between $\cup_i F_i$ and $\cup_i F_i$.

Link signatures

In $H_2(D^4 \smallsetminus \cup_i F_i; \mathbb{C}_\mu)$ there is a Hermitian intersection form.

Theorem. *Its signature* $\sigma_\zeta(L)$ *does not depend on* $F_1, \dots, F_m$ *.*

Proof. Any F_i' with $\partial F_i' = F_i' \cap \partial D^4 = l_i$ is cobordant to F_i . The cobordisms $W_i \subset D^4 \times I$ can be made pairwise transversal. They define a cobordism $D^4 \times I \smallsetminus \cup_i N(W_i)$ between $D^4 \smallsetminus \cup_i N(F_i)$ and $D^4 \smallsetminus \cup_i N(F_i')$. The boundary of the cobordism consists of $D^4 \smallsetminus \cup_i N(F_i)$, $D^4 \smallsetminus \cup_i N(F_i')$ and a homologically negligible part $\partial(N(\cup_i W_i))$, the boundary of a regular neighborhood of the cobordism $\cup_i W_i$ between $\cup_i F_i$ and $\cup_i F_i$. Hence, $\sigma(D^4 \smallsetminus \cup_i F_i) = \sigma(D^4 \smallsetminus \cup_i F_i')$. $\square$

Link signatures

In $H_2(D^4 \setminus \cup_i F_i; \mathbb{C}_\mu)$ there is a Hermitian intersection form.

Theorem. *Its signature* $\sigma_\zeta(L)$ *does not depend on* $F_1, \dots, F_m$*.*

$\square$

The same arguments work for $L = \cup_{i=1}^m L_i$, where L_i are oriented submanifolds of codimension 2 of S^{2n-1} transversal to each other, and F_i are submanifolds of D^{2n} transversal to each other.

Link signatures

In $H_2(D^4 \smallsetminus \cup_i F_i; \mathbb{C}_\mu)$ there is a Hermitian intersection form.

Theorem. *Its signature* $\sigma_\zeta(L)$ *does not depend on* $F_1, \dots, F_m$ *.*

$\square$

The same arguments work for $L = \cup_{i=1}^m L_i$, where L_i are oriented submanifolds of codimension 2 of S^{2n-1} transversal to each other, and F_i are submanifolds of D^{2n} transversal to each other.

If n is odd, then the intersection form in $H_n(D^{2n} \smallsetminus \cup_i F_i; \mathbb{C}_\mu)$ is skew-Hermitian.

Link signatures

In $H_2(D^4 \setminus \cup_i F_i; \mathbb{C}_\mu)$ there is a Hermitian intersection form.

Theorem. *Its signature* $\sigma_\zeta(L)$ *does not depend on* $F_1, \dots, F_m$ *.*
□

The same arguments work for $L = \cup_{i=1}^m L_i$, where L_i are oriented submanifolds of codimension 2 of S^{2n-1} transversal to each other, and F_i are submanifolds of D^{2n} transversal to each other.

If n is odd, then the intersection form in $H_n(D^{2n} \setminus \cup_i F_i; \mathbb{C}_\mu)$ is skew-Hermitian. Multiply it by $i = \sqrt{-1}$ and denote the signature of the Hermitian form by $\sigma_\zeta(L)$.

Digression on higher dim links.

There is a spectrum of objects considered generalizations of classical knots and links.

Digression on higher dim links.

There is a spectrum of objects considered generalizations of classical knots and links.

The closest generalization of classical knots are pairs (S^n, K), where K is a smooth submanifold diffeomorphic to S^{n-2}.

Digression on higher dim links.

There is a spectrum of objects considered generalizations of classical knots and links.
The closest generalization of classical knots are pairs (S^n, K), where K is a smooth submanifold diffeomorphic to S^{n-2}.

Then the requirements on K are weakened.

Digression on higher dim links.

There is a spectrum of objects considered generalizations of classical knots and links.

The closest generalization of classical knots are pairs (S^n, K), where K is a smooth submanifold diffeomorphic to S^{n-2}.

One may require K to be only homeomorphic to S^{n-2}, not diffeomorphic

Digression on higher dim links.

There is a spectrum of objects considered generalizations of classical knots and links.
The closest generalization of classical knots are pairs (S^n, K) , where K is a smooth submanifold diffeomorphic to S^{n-2} .

One may require K to be only homeomorphic to S^{n-2} , not diffeomorphic ,
or just a homology sphere of dimension $n-2$

Digression on higher dim links.

There is a spectrum of objects considered generalizations of classical knots and links.
The closest generalization of classical knots are pairs (S^n, K) , where K is a smooth submanifold diffeomorphic to S^{n-2} .

One may require K to be only homeomorphic to S^{n-2} , not diffeomorphic ,
or just a homology sphere of dimension $n-2$,
or a submanifold of dimension $n-2$ with $S^n \setminus K$ fibered over S^1 .

Digression on higher dim links.

There is a spectrum of objects considered generalizations of classical knots and links.
The closest generalization of classical knots are pairs (S^n, K), where K is a smooth submanifold diffeomorphic to S^{n-2}.

The codimension two is most important.

Digression on higher dim links.

There is a spectrum of objects considered generalizations of classical knots and links.
The closest generalization of classical knots are pairs (S^n, K), where K is a smooth submanifold diffeomorphic to S^{n-2}.

The closest higher-dimensional counter-part of classical links are pairs (S^n, L), where L is a collection of its disjoint smooth submanifolds diffeomorphic to S^{n-2}.

Digression on higher dim links.

There is a spectrum of objects considered generalizations of classical knots and links.
The closest generalization of classical knots are pairs (S^n, K), where K is a smooth submanifold diffeomorphic to S^{n-2}.

The closest higher-dimensional counter-part of classical links are pairs (S^n, L), where L is a collection of its disjoint smooth submanifolds diffeomorphic to S^{n-2}.

Then the restrictions to submanifolds are weakened.

Digression on higher dim links.

There is a spectrum of objects considered generalizations of classical knots and links.
The closest generalization of classical knots are pairs (S^n, K), where K is a smooth submanifold diffeomorphic to S^{n-2}.

The closest higher-dimensional counter-part of classical links are pairs (S^n, L), where L is a collection of its disjoint smooth submanifolds diffeomorphic to S^{n-2}.

Then the restrictions to submanifolds are weakened.

but they are usually required to be disjoint.

Digression on higher dim links.

There is a spectrum of objects considered generalizations of classical knots and links.
The closest generalization of classical knots are pairs (S^n, K), where K is a smooth submanifold diffeomorphic to S^{n-2}.

I suggest to allow transversal intersections of the submanifolds.

Digression on higher dim links.

There is a spectrum of objects considered generalizations of classical knots and links.
The closest generalization of classical knots are pairs (S^n, K), where K is a smooth submanifold diffeomorphic to S^{n-2}.

I suggest to allow transversal intersections of the submanifolds.
I can prove something in this situation.

Digression on higher dim links.

There is a spectrum of objects considered generalizations of classical knots and links.
The closest generalization of classical knots are pairs (S^n, K), where K is a smooth submanifold diffeomorphic to S^{n-2}.

I suggest to allow transversal intersections of the submanifolds.
Other reasons:

Digression on higher dim links.

There is a spectrum of objects considered generalizations of classical knots and links.
The closest generalization of classical knots are pairs (S^n, K), where K is a smooth submanifold diffeomorphic to S^{n-2}.

I suggest to allow transversal intersections of the submanifolds.
Other reasons:
1. In the classical dimension it is easy to be disjoint.

Digression on higher dim links.

There is a spectrum of objects considered generalizations of classical knots and links.
The closest generalization of classical knots are pairs (S^n, K), where K is a smooth submanifold diffeomorphic to S^{n-2}.

I suggest to allow transversal intersections of the submanifolds.
Other reasons:
1. In the classical dimension it is easy to be disjoint. Generic submanifolds of codimension 2 in a manifold of dimension > 3 intersect.

Digression on higher dim links.

There is a spectrum of objects considered generalizations of classical knots and links.
The closest generalization of classical knots are pairs (S^n, K), where K is a smooth submanifold diffeomorphic to S^{n-2}.

I suggest to allow transversal intersections of the submanifolds.
Other reasons:
1. In the classical dimension it is easy to be disjoint. Generic submanifolds of codimension 2 in a manifold of dimension > 3 intersect.
2. A link of an algebraic hypersurface $H \subset \mathbb{C}^n$ with $n \geq 3$ cannot be a union of disjoint submanifolds.

Estimates of twisted homology

Sometimes one may want to get rid of twisted homology.

Estimates of twisted homology

Sometimes one may want to get rid of twisted homology.

If twisted homology does not vanish itself, it may be desirable to find something larger, but better understood.

Estimates of twisted homology

Sometimes one may want to get rid of twisted homology.

If twisted homology does not vanish itself, it may be desirable to find something larger, but better understood.

We will show that often the dimensions of twisted homology are estimated by the dimensions of untwisted ones.

Estimates of twisted homology

Lemma 1. (The principal algebraic lemma of the Morse theory.) For a complex $C : \dots \longrightarrow C_i \xrightarrow{\partial_i} C_{i-1} \longrightarrow$ of finite dimensional vector spaces over a field F

$$\sum_{s=r}^{2n+r} (-1)^{s-r} \dim_F H_s(C) = \\ = \sum_{s=r}^{2n+r} (-1)^{s-r} \dim_F C_s - \operatorname{rk} \partial_{r-1} - \operatorname{rk} \partial_{2n+r}.$$

Estimates of twisted homology

Lemma 1. (The principal algebraic lemma of the Morse theory.) For a complex $C : \dots \longrightarrow C_i \xrightarrow{\partial_i} C_{i-1} \longrightarrow$ of finite dimensional vector spaces over a field F

$$\sum_{s=r}^{2n+r} (-1)^{s-r} \dim_F H_s(C) = \\ = \sum_{s=r}^{2n+r} (-1)^{s-r} \dim_F C_s - \operatorname{rk} \partial_{r-1} - \operatorname{rk} \partial_{2n+r}.$$

Proof. For $n = 0$: Since $H_s(C) = \operatorname{Ker} \partial_s / \operatorname{Im} \partial_{s+1}$, we have $\dim H_s(C) = \dim \operatorname{Ker} \partial_s - \dim \operatorname{Im} \partial_{s+1}$.

Estimates of twisted homology

Lemma 1. (The principal algebraic lemma of the Morse theory.) For a complex $C : \dots \to C_i \xrightarrow{\partial_i} C_{i-1} \to$ of finite dimensional vector spaces over a field F

$$\sum_{s=r}^{2n+r} (-1)^{s-r} \dim_F H_s(C) = \\ = \sum_{s=r}^{2n+r} (-1)^{s-r} \dim_F C_s - \operatorname{rk} \partial_{r-1} - \operatorname{rk} \partial_{2n+r}.$$

Proof. For $n = 0$: Since $H_s(C) = \operatorname{Ker} \partial_s / \operatorname{Im} \partial_{s+1}$, we have $\dim H_s(C) = \dim \operatorname{Ker} \partial_s - \dim \operatorname{Im} \partial_{s+1}$.
Further, $\dim \operatorname{Im} \partial_{s+1} = \operatorname{rk} \partial_{s+1}$, and
$\dim \operatorname{Ker} \partial_s = \dim C_s - \operatorname{rk} \partial_s$. Hence,
$\dim H_s = \dim C_s - \operatorname{rk} \partial_s - \operatorname{rk} \partial_{s+1}$

Estimates of twisted homology

Lemma 1. (The principal algebraic lemma of the Morse theory.) For a complex $C : \dots \to C_i \xrightarrow{\partial_i} C_{i-1} \to$ of finite dimensional vector spaces over a field F

$$\sum_{s=r}^{2n+r} (-1)^{s-r} \dim_F H_s(C) = \\ = \sum_{s=r}^{2n+r} (-1)^{s-r} \dim_F C_s - \operatorname{rk} \partial_{r-1} - \operatorname{rk} \partial_{2n+r}.$$

Proof. For $n = 0$: Since $H_s(C) = \operatorname{Ker} \partial_s / \operatorname{Im} \partial_{s+1}$, we have $\dim H_s(C) = \dim \operatorname{Ker} \partial_s - \dim \operatorname{Im} \partial_{s+1}$.
Further, $\dim \operatorname{Im} \partial_{s+1} = \operatorname{rk} \partial_{s+1}$, and $\dim \operatorname{Ker} \partial_s = \dim C_s - \operatorname{rk} \partial_s$. Hence,
$\dim H_s = \dim C_s - \operatorname{rk} \partial_s - \operatorname{rk} \partial_{s+1}$
In general case, make alternating summation of this for $s = r, \dots, 2n + s$. □

Estimates of twisted homology

Lemma 1. (The principal algebraic lemma of the Morse theory.) For a complex $C : \cdots \to C_i \xrightarrow{\partial_i} C_{i-1} \to$ of finite dimensional vector spaces over a field F

$$\sum_{s=r}^{2n+r} (-1)^{s-r} \dim_F H_s(C) = \\ = \sum_{s=r}^{2n+r} (-1)^{s-r} \dim_F C_s - \operatorname{rk} \partial_{r-1} - \operatorname{rk} \partial_{2n+r}.$$

□

Lemma 2. Let P and Q be fields, $R \subset Q$ a subring and $h : R \to P$ a ring homomorphism.

Estimates of twisted homology

Lemma 1. (The principal algebraic lemma of the Morse theory.) For a complex $C : \dots \to C_i \xrightarrow{\partial_i} C_{i-1} \to$ of finite dimensional vector spaces over a field F

$$\sum_{s=r}^{2n+r}(-1)^{s-r}\dim_F H_s(C) = \\ = \sum_{s=r}^{2n+r}(-1)^{s-r}\dim_F C_s - \operatorname{rk}\partial_{r-1} - \operatorname{rk}\partial_{2n+r}. \qquad \square$$

Lemma 2. Let P and Q be fields, $R \subset Q$ a subring and $h : R \to P$ a ring homomorphism. Let $C : \dots \to C_p \to C_{p-1} \to \dots \to C_1 \to C_0$ be a complex of free finitely generated R-modules. Then for any n and r

$$\sum_{s=r}^{2n+r}(-1)^{s-r}\dim_Q H_s(C \otimes_R Q) \\ \leq \sum_{s=r}^{2n+r}(-1)^{s-r}\dim_P H_s(C \otimes_h P)\,. \qquad \square$$

Estimates of twisted homology

Lemma 1. (The principal algebraic lemma of the Morse theory.) For a complex $C : \dots \to C_i \xrightarrow{\partial_i} C_{i-1} \to$ of finite dimensional vector spaces over a field F

$$\sum_{s=r}^{2n+r}(-1)^{s-r}\dim_F H_s(C) = \\ = \sum_{s=r}^{2n+r}(-1)^{s-r}\dim_F C_s - \operatorname{rk}\partial_{r-1} - \operatorname{rk}\partial_{2n+r}. \qquad \square$$

Lemma 2. Let P and Q be fields, $R \subset Q$ a subring and $h : R \to P$ a ring homomorphism. Let $C : \dots \to C_p \to C_{p-1} \to \dots \to C_1 \to C_0$ be a complex of free finitely generated R-modules. Then for any n and r

$$\sum_{s=r}^{2n+r}(-1)^{s-r}\dim_Q H_s(C \otimes_R Q) \\ \leq \sum_{s=r}^{2n+r}(-1)^{s-r}\dim_P H_s(C \otimes_h P).$$

Proof. $\dim_Q C_i \otimes RQ = \dim_P C_i \otimes hP$, $\operatorname{rk}\partial_i^Q \geq \operatorname{rk}\partial_i^P$. $\square$

Estimates of twisted homology

Lemma 2. Let P and Q be fields, $R \subset Q$ a subring and $h : R \to P$ a ring homomorphism. Let $C : \cdots \to C_p \to C_{p-1} \to \cdots \to C_1 \to C_0$ be a complex of free finitely generated R-modules. Then for any n and r

$$\sum_{s=r}^{2n+r} (-1)^{s-r} \dim_Q H_s(C \otimes_R Q) \leq \sum_{s=r}^{2n+r} (-1)^{s-r} \dim_P H_s(C \otimes_h P).$$

Theorem. Let X be a finite cw-complex, $\mu : H_1(X) \to \mathbb{C}^\times$ a homomorphism. If $\operatorname{Im} \mu \subset \mathbb{C}^\times$ generates a subring R of $\mathbb{C}$ and there is a ring homomorphism $h : R \to P$, where P is a field, such that $h\mu(H_1(X)) = 1$, then

$$\sum_{s=r}^{2n+r} (-1)^{s-r} \dim H_s(X; \mathbb{C}_\mu) \leq \sum_{s=r}^{2n+r} (-1)^{s-r} \dim_P H_s(X; P).$$

□

Estimates of twisted homology

Theorem. Let X be a finite cw-complex, $\mu : H_1(X) \to \mathbb{C}^\times$ a homomorphism. If $\operatorname{Im} \mu \subset \mathbb{C}^\times$ generates a subring R of $\mathbb{C}$ and there is a ring homomorphism $h : R \to P$, where P is a field, such that $h\mu(H_1(X)) = 1$, then

$$\sum_{s=r}^{2n+r} (-1)^{s-r} \dim H_s(X; \mathbb{C}_\mu) \leq \sum_{s=r}^{2n+r} (-1)^{s-r} \dim_P H_s(X; P).$$

□

Special cases: 1. $H_1(X)$ is generated by g, $\mu(g)$ is an algebraic number, f is the minimal integer polynomial with relatively prime coefficients which annihilates $\mu(g)$.

Estimates of twisted homology

Theorem. Let X be a finite cw-complex, $\mu : H_1(X) \to \mathbb{C}^\times$ a homomorphism. If $\operatorname{Im}\mu \subset \mathbb{C}^\times$ generates a subring R of $\mathbb{C}$ and there is a ring homomorphism $h : R \to P$, where P is a field, such that $h\mu(H_1(X)) = 1$, then

$$\sum_{s=r}^{2n+r} (-1)^{s-r} \dim H_s(X; \mathbb{C}_\mu) \le \sum_{s=r}^{2n+r} (-1)^{s-r} \dim_P H_s(X; P) .$$ □

Special cases: 1. $H_1(X)$ is generated by g,
$\mu(g)$ is an algebraic number,
f is the minimal integer polynomial with relatively prime coefficients which annihilates $\mu(g)$.
Assume p is a prime number which divides $f(1)$.

Estimates of twisted homology

Theorem. Let X be a finite cw-complex, $\mu : H_1(X) \to \mathbb{C}^\times$ a homomorphism. If $\operatorname{Im}\mu \subset \mathbb{C}^\times$ generates a subring R of $\mathbb{C}$ and there is a ring homomorphism $h : R \to P$, where P is a field, such that $h\mu(H_1(X)) = 1$, then

$$\sum_{s=r}^{2n+r}(-1)^{s-r}\dim H_s(X;\mathbb{C}_\mu) \le \sum_{s=r}^{2n+r}(-1)^{s-r}\dim_P H_s(X;P).$$

□

Special cases: 1. $H_1(X)$ is generated by g,

$\mu(g)$ is an algebraic number,

f is the minimal integer polynomial with relatively prime coefficients which annihilates $\mu(g)$.

Assume p is a prime number which divides $f(1)$.

Then $R = \mathbb{Z}[\mu(g)]$, $P = \mathbb{Z}/p$ and

$h : \mathbb{Z}[\mu(g)] \to \mathbb{Z}/p,\ \ \mu(g) \mapsto 1$.

Estimates of twisted homology

Theorem. Let X be a finite cw-complex, $\mu : H_1(X) \to \mathbb{C}^\times$ a homomorphism. If $\operatorname{Im}\mu \subset \mathbb{C}^\times$ generates a subring R of $\mathbb{C}$ and there is a ring homomorphism $h : R \to P$, where P is a field, such that $h\mu(H_1(X)) = 1$, then

$$\sum_{s=r}^{2n+r}(-1)^{s-r}\dim H_s(X;\mathbb{C}_\mu) \leq \sum_{s=r}^{2n+r}(-1)^{s-r}\dim_P H_s(X;P)\,. \qquad \square$$

Special cases: 2. $H_1(X)$ is generated by $g_1, \dots, g_k$, $\mu(g_1), \dots, \mu(g_k)$ are algebraic numbers, f_i is the minimal integer polynomial with relatively prime coefficients which annihilates $\mu(g_i)$.

Estimates of twisted homology

Theorem. Let X be a finite cw-complex, $\mu : H_1(X) \to \mathbb{C}^\times$ a homomorphism. If $\operatorname{Im}\mu \subset \mathbb{C}^\times$ generates a subring R of $\mathbb{C}$ and there is a ring homomorphism $h : R \to P$, where P is a field, such that $h\mu(H_1(X)) = 1$, then

$$\sum_{s=r}^{2n+r} (-1)^{s-r} \dim H_s(X; \mathbb{C}_\mu) \le \sum_{s=r}^{2n+r} (-1)^{s-r} \dim_P H_s(X; P) .$$ □

Special cases: 2. $H_1(X)$ is generated by $g_1, \dots, g_k$, $\mu(g_1), \dots, \mu(g_k)$ are algebraic numbers,
f_i is the minimal integer polynomial with relatively prime coefficients which annihilates $\mu(g_i)$.
Assume p is a prime number which divides $f_1(1), \dots, f_k(1)$.

Estimates of twisted homology

Theorem. Let X be a finite cw-complex, $\mu : H_1(X) \to \mathbb{C}^\times$ a homomorphism. If $\operatorname{Im}\mu \subset \mathbb{C}^\times$ generates a subring R of $\mathbb{C}$ and there is a ring homomorphism $h : R \to P$, where P is a field, such that $h\mu(H_1(X)) = 1$, then

$$\sum_{s=r}^{2n+r}(-1)^{s-r}\dim H_s(X;\mathbb{C}_\mu) \le \sum_{s=r}^{2n+r}(-1)^{s-r}\dim_P H_s(X;P).$$ □

Special cases: 2. $H_1(X)$ is generated by $g_1,\dots,g_k$, $\mu(g_1),\dots,\mu(g_k)$ are algebraic numbers,
f_i is the minimal integer polynomial with relatively prime coefficients which annihilates $\mu(g_i)$.
Assume p is a prime number which divides $f_1(1),\dots,f_k(1)$.
Then $R = \mathbb{Z}[\mu(g_1),\dots,\mu(g_k)]$, $P = \mathbb{Z}/p$ and
$h : \mathbb{Z}[\mu(g_1),\dots\mu(g_k)] \to \mathbb{Z}/p,\ \ \mu(g_i) \mapsto 1$.

Estimates of twisted homology

Theorem. Let X be a finite cw-complex, $\mu : H_1(X) \to \mathbb{C}^\times$ a homomorphism. If $\operatorname{Im}\mu \subset \mathbb{C}^\times$ generates a subring R of $\mathbb{C}$ and there is a ring homomorphism $h : R \to P$, where P is a field, such that $h\mu(H_1(X)) = 1$, then

$$\sum_{s=r}^{2n+r}(-1)^{s-r}\dim H_s(X;\mathbb{C}_\mu) \leq \sum_{s=r}^{2n+r}(-1)^{s-r}\dim_P H_s(X;P).$$ □

Special cases: 3. $H_1(X)$ is generated by g, $\mu(g)$ is a transcendental number.

Estimates of twisted homology

Theorem. Let X be a finite cw-complex, $\mu : H_1(X) \to \mathbb{C}^\times$ a homomorphism. If $\operatorname{Im}\mu \subset \mathbb{C}^\times$ generates a subring R of $\mathbb{C}$ and there is a ring homomorphism $h : R \to P$, where P is a field, such that $h\mu(H_1(X)) = 1$, then

$$\sum_{s=r}^{2n+r}(-1)^{s-r}\dim H_s(X;\mathbb{C}_\mu) \le \sum_{s=r}^{2n+r}(-1)^{s-r}\dim_P H_s(X;P)\,. \qquad \square$$

Special cases: 3. $H_1(X)$ is generated by g, $\mu(g)$ is a transcendental number.
Then $R = \mathbb{Z}[\mu(g)]$, $P = \mathbb{Q}$ and
$h : \mathbb{Z}[\mu(g)] \to \mathbb{Q},\ \ \mu(g) \mapsto 1$.

Estimates of twisted homology

Theorem. Let X be a finite cw-complex, $\mu : H_1(X) \to \mathbb{C}^\times$ a homomorphism. If $\operatorname{Im}\mu \subset \mathbb{C}^\times$ generates a subring R of $\mathbb{C}$ and there is a ring homomorphism $h : R \to P$, where P is a field, such that $h\mu(H_1(X)) = 1$, then

$$\sum_{s=r}^{2n+r}(-1)^{s-r}\dim H_s(X;\mathbb{C}_\mu) \le \sum_{s=r}^{2n+r}(-1)^{s-r}\dim_P H_s(X;P)\,. \qquad \square$$

Special cases: 3. $H_1(X)$ is generated by g, $\mu(g)$ is a transcendental number.
Then $R = \mathbb{Z}[\mu(g)]$, $P = \mathbb{Q}$ and
$h : \mathbb{Z}[\mu(g)] \to \mathbb{Q}, \ \ \mu(g) \mapsto 1$.

For generic $\mu(g)$ twisted homology are not greater than untwisted.

Estimates of twisted homology

Theorem. Let X be a finite cw-complex, $\mu : H_1(X) \to \mathbb{C}^\times$ a homomorphism. If $\operatorname{Im}\mu \subset \mathbb{C}^\times$ generates a subring R of $\mathbb{C}$ and there is a ring homomorphism $h : R \to P$, where P is a field, such that $h\mu(H_1(X)) = 1$, then

$$\sum_{s=r}^{2n+r}(-1)^{s-r}\dim H_s(X;\mathbb{C}_\mu) \le \sum_{s=r}^{2n+r}(-1)^{s-r}\dim_P H_s(X;P)\,. \qquad \square$$

Theorem. $H_1(X)$ is generated by $g_1,\dots,g_k$, $\mu,\nu : H_1(X) \to \mathbb{C}^\times$ be homomorphisms, $\mu(g_1),\dots,\mu(g_k),\nu(g_1),\dots,\nu(g_k)$ be transcendental numbers such that $(\mu(g_1),\dots,\mu(g_k)) \in \mathbb{C}^k$ is a general point of a variety V and $(\nu(g_1),\dots,\nu(g_k)) \in \mathbb{C}^k$ is a general point of a subvariety $W \subset V$. Then

$$\sum_{s=r}^{2n+r}(-1)^{s-r}\dim H_s(X;\mathbb{C}_\mu) \le \sum_{s=r}^{2n+r}(-1)^{s-r}\dim H_s(X;\mathbb{C}_\nu)\,. \qquad \square$$

Span inequalities

Let $L_1, \ldots, L_m \subset S^{2n-1}$ be smooth oriented transversal to each other submanifolds of codimension 2, $L = L_1 \cup \cdots \cup L_m$.

Span inequalities

Let $L_1, \ldots, L_m \subset S^{2n-1}$ be smooth oriented transversal to each other submanifolds of codimension 2, $L = L_1 \cup \cdots \cup L_m$.

Let $\zeta_i \in \mathbb{C}$ be algebraic numbers with $|\zeta_i| = 1$, and f_i be irreducible integer polynomials with $f_i(\zeta_i) = 0$.

Span inequalities

Let $L_1, \dots, L_m \subset S^{2n-1}$ be smooth oriented transversal to each other submanifolds of codimension 2, $L = L_1 \cup \dots \cup L_m$.

Let $\zeta_i \in \mathbb{C}$ be algebraic numbers with $|\zeta_i| = 1$, and f_i be irreducible integer polynomials with $f_i(\zeta_i) = 0$.

Suppose prime number p divides $f_i(1)$ for $i = 1, \dots, m$.

Span inequalities

Let $L_1, \dots, L_m \subset S^{2n-1}$ be smooth oriented transversal to each other submanifolds of codimension 2, $L = L_1 \cup \dots \cup L_m$.
Let $\zeta_i \in \mathbb{C}$ be algebraic numbers with $|\zeta_i| = 1$, and f_i be irreducible integer polynomials with $f_i(\zeta_i) = 0$.
Suppose prime number p divides $f_i(1)$ for $i = 1, \dots, m$.
Let $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ take a meridian of L_i to ζ_i.

Span inequalities

Let $L_1, \dots, L_m \subset S^{2n-1}$ be smooth oriented transversal to each other submanifolds of codimension 2, $L = L_1 \cup \dots \cup L_m$.

Let $\zeta_i \in \mathbb{C}$ be algebraic numbers with $|\zeta_i| = 1$, and f_i be irreducible integer polynomials with $f_i(\zeta_i) = 0$.
Suppose prime number p divides $f_i(1)$ for $i = 1, \dots, m$.
Let $\mu : \pi_1(S^{2n-1} \setminus L) \to \mathbb{C}^\times$ take a meridian of L_i to ζ_i .
Let $F_i \subset D^{2n}$ be oriented compact smooth submanifolds transversal to each other, with $\partial F_i = F_i \cap \partial D^{2n} = L_i$.

Span inequalities

Let $L_1, \dots, L_m \subset S^{2n-1}$ be smooth oriented transversal to each other submanifolds of codimension 2, $L = L_1 \cup \dots \cup L_m$.

Let $\zeta_i \in \mathbb{C}$ be algebraic numbers with $|\zeta_i| = 1$, and f_i be irreducible integer polynomials with $f_i(\zeta_i) = 0$.

Suppose prime number p divides $f_i(1)$ for $i = 1, \dots, m$.

Let $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ take a meridian of L_i to ζ_i.

Let $F_i \subset D^{2n}$ be oriented compact smooth submanifolds transversal to each other, with $\partial F_i = F_i \cap \partial D^{2n} = L_i$.

Put $F = \cup_i F_i$. Extend $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ to $\mu : \pi_1(D^{2n} \smallsetminus F) \to \mathbb{C}^\times$.

Span inequalities

Let $L_1, \dots, L_m \subset S^{2n-1}$ be smooth oriented transversal to each other submanifolds of codimension 2, $L = L_1 \cup \dots \cup L_m$.

Let $\zeta_i \in \mathbb{C}$ be algebraic numbers with $|\zeta_i| = 1$, and f_i be irreducible integer polynomials with $f_i(\zeta_i) = 0$.
Suppose prime number p divides $f_i(1)$ for $i = 1, \dots, m$.
Let $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ take a meridian of L_i to ζ_i.
Let $F_i \subset D^{2n}$ be oriented compact smooth submanifolds transversal to each other, with $\partial F_i = F_i \cap \partial D^{2n} = L_i$.
Put $F = \cup_i F_i$. Extend $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ to $\mu : \pi_1(D^{2n} \smallsetminus F) \to \mathbb{C}^\times$.

Obviously,

$$|\sigma_\zeta(L)| \le \dim H_n(D^{2n} \smallsetminus F; \mathbb{C}_\mu) \le H_n(D^{2n} \smallsetminus F; \mathbb{Z}/p)$$

Span inequalities

Let $L_1, \dots, L_m \subset S^{2n-1}$ be smooth oriented transversal to each other submanifolds of codimension 2, $L = L_1 \cup \dots \cup L_m$.

Let $\zeta_i \in \mathbb{C}$ be algebraic numbers with $|\zeta_i| = 1$, and f_i be irreducible integer polynomials with $f_i(\zeta_i) = 0$.
Suppose prime number p divides $f_i(1)$ for $i = 1, \dots, m$.
Let $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ take a meridian of L_i to ζ_i.
Let $F_i \subset D^{2n}$ be oriented compact smooth submanifolds transversal to each other, with $\partial F_i = F_i \cap \partial D^{2n} = L_i$.
Put $F = \cup_i F_i$. Extend $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ to $\mu : \pi_1(D^{2n} \smallsetminus F) \to \mathbb{C}^\times$.

Obviously,

$$|\sigma_\zeta(L)| \leq \dim H_n(D^{2n} \smallsetminus F; \mathbb{C}_\mu) \leq H_n(D^{2n} \smallsetminus F; \mathbb{Z}/p)$$
$$= \dim H_{n-1}(F; \mathbb{Z}/p).$$

Span inequalities

Let $L_1, \dots, L_m \subset S^{2n-1}$ be smooth oriented transversal to each other submanifolds of codimension 2, $L = L_1 \cup \dots \cup L_m$.

Let $\zeta_i \in \mathbb{C}$ be algebraic numbers with $|\zeta_i| = 1$, and f_i be irreducible integer polynomials with $f_i(\zeta_i) = 0$. Suppose prime number p divides $f_i(1)$ for $i = 1, \dots, m$. Let $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ take a meridian of L_i to ζ_i. Let $F_i \subset D^{2n}$ be oriented compact smooth submanifolds transversal to each other, with $\partial F_i = F_i \cap \partial D^{2n} = L_i$. Put $F = \cup_i F_i$. Extend $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ to $\mu : \pi_1(D^{2n} \smallsetminus F) \to \mathbb{C}^\times$.

Obviously,

$|\sigma_\zeta(L)| \le \dim H_n(D^{2n} \smallsetminus F; \mathbb{C}_\mu) \le H_n(D^{2n} \smallsetminus F; \mathbb{Z}/p)$
$= \dim H_{n-1}(F; \mathbb{Z}/p)$. Thus, $|\sigma_\zeta(L)| \le \dim H_{n-1}(F; \mathbb{Z}/p)$.

- Twisted homology
- Duality
- Unitary local coefficients
- Signatures
- Link signatures
- Digression on higher dim links.
- Estimates of twisted homology
- Span inequalities
- Slice inequalities

Span inequalities

Let $L_1, \dots, L_m \subset S^{2n-1}$ be smooth oriented transversal to each other submanifolds of codimension 2, $L = L_1 \cup \dots \cup L_m$.

Let $\zeta_i \in \mathbb{C}$ be algebraic numbers with $|\zeta_i| = 1$, and f_i be irreducible integer polynomials with $f_i(\zeta_i) = 0$. Suppose prime number p divides $f_i(1)$ for $i = 1, \dots, m$. Let $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ take a meridian of L_i to ζ_i. Let $F_i \subset D^{2n}$ be oriented compact smooth submanifolds transversal to each other, with $\partial F_i = F_i \cap \partial D^{2n} = L_i$. Put $F = \cup_i F_i$. Extend $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ to $\mu : \pi_1(D^{2n} \smallsetminus F) \to \mathbb{C}^\times$.

Obviously,

$|\sigma_\zeta(L)| \leq \dim H_n(D^{2n} \smallsetminus F; \mathbb{C}_\mu) \leq H_n(D^{2n} \smallsetminus F; \mathbb{Z}/p)$ $= \dim H_{n-1}(F; \mathbb{Z}/p)$. Thus, $|\sigma_\zeta(L)| \leq \dim H_{n-1}(F; \mathbb{Z}/p)$.

Similarly one can prove:

Span inequalities

Let $L_1, \dots, L_m \subset S^{2n-1}$ be smooth oriented transversal to each other submanifolds of codimension 2, $L = L_1 \cup \dots \cup L_m$.

Let $\zeta_i \in \mathbb{C}$ be algebraic numbers with $|\zeta_i| = 1$, and f_i be irreducible integer polynomials with $f_i(\zeta_i) = 0$.

Suppose prime number p divides $f_i(1)$ for $i = 1, \dots, m$.

Let $\mu : \pi_1(S^{2n-1} \setminus L) \to \mathbb{C}^\times$ take a meridian of L_i to ζ_i.

Let $F_i \subset D^{2n}$ be oriented compact smooth submanifolds transversal to each other, with $\partial F_i = F_i \cap \partial D^{2n} = L_i$.

Put $F = \cup_i F_i$. Extend $\mu : \pi_1(S^{2n-1} \setminus L) \to \mathbb{C}^\times$ to $\mu : \pi_1(D^{2n} \setminus F) \to \mathbb{C}^\times$.

Theorem. For any integer r with $0 \le r \le \frac{n}{2}$

Span inequalities

Let $L_1, \dots, L_m \subset S^{2n-1}$ be smooth oriented transversal to each other submanifolds of codimension 2, $L = L_1 \cup \dots \cup L_m$.

Let $\zeta_i \in \mathbb{C}$ be algebraic numbers with $|\zeta_i| = 1$, and f_i be irreducible integer polynomials with $f_i(\zeta_i) = 0$.
Suppose prime number p divides $f_i(1)$ for $i = 1, \dots, m$.
Let $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ take a meridian of L_i to ζ_i.
Let $F_i \subset D^{2n}$ be oriented compact smooth submanifolds transversal to each other, with $\partial F_i = F_i \cap \partial D^{2n} = L_i$.
Put $F = \cup_i F_i$. Extend $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ to $\mu : \pi_1(D^{2n} \smallsetminus F) \to \mathbb{C}^\times$.

Theorem. For any integer r with $0 \le r \le \frac{n}{2}$

$$|\sigma_\zeta(L)| + \sum_{s=0}^{2r}(-1)^s \dim H_{r-1-s}(S^{2n-1} \smallsetminus L; \mathbb{C}_\zeta) \le \sum_{s=0}^{2r}(-1)^s \dim H_{n-1+s}(F, L; \mathbb{Z}/p) + \sum_{s=0}^{2r}(-1)^s \dim H_{n-2-s}(F, L; \mathbb{Z}/p)$$

Span inequalities

Let $L_1, \dots, L_m \subset S^{2n-1}$ be smooth oriented transversal to each other submanifolds of codimension 2, $L = L_1 \cup \dots \cup L_m$.

Let $\zeta_i \in \mathbb{C}$ be algebraic numbers with $|\zeta_i| = 1$, and f_i be irreducible integer polynomials with $f_i(\zeta_i) = 0$.

Suppose prime number p divides $f_i(1)$ for $i = 1, \dots, m$.

Let $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ take a meridian of L_i to ζ_i.

Let $F_i \subset D^{2n}$ be oriented compact smooth submanifolds transversal to each other, with $\partial F_i = F_i \cap \partial D^{2n} = L_i$.

The r th nullity $n^r_\zeta(L)$ is defined as

$\sum_{s=0}^{2r} (-1)^s \dim H_{n+s}(S^{2n-1} \smallsetminus \cup_{i=1}^m L_i; \mathbb{C}_\mu)$.

Span inequalities

Let $L_1, \dots, L_m \subset S^{2n-1}$ be smooth oriented transversal to each other submanifolds of codimension 2, $L = L_1 \cup \dots \cup L_m$.

Let $\zeta_i \in \mathbb{C}$ be algebraic numbers with $|\zeta_i| = 1$, and f_i be irreducible integer polynomials with $f_i(\zeta_i) = 0$. Suppose prime number p divides $f_i(1)$ for $i = 1, \dots, m$. Let $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ take a meridian of L_i to ζ_i . Let $F_i \subset D^{2n}$ be oriented compact smooth submanifolds transversal to each other, with $\partial F_i = F_i \cap \partial D^{2n} = L_i$.

Theorem. For any integer r with $0 \le r \le \frac{n}{2}$

$$|\sigma_\zeta(L)| + n^r_\zeta(L) \le \sum_{s=0}^{2r} (-1)^s \dim H_{n-1+s}(F, L; \mathbb{Z}/p)$$
$$+ \sum_{s=0}^{2r} (-1)^s \dim H_{n-2-s}(F, L; \mathbb{Z}/p)$$

Span inequalities

Let $L_1, \dots, L_m \subset S^{2n-1}$ be smooth oriented transversal to each other submanifolds of codimension 2, $L = L_1 \cup \dots \cup L_m$.

Let $\zeta_i \in \mathbb{C}$ be algebraic numbers with $|\zeta_i| = 1$, and f_i be irreducible integer polynomials with $f_i(\zeta_i) = 0$.
Suppose prime number p divides $f_i(1)$ for $i = 1, \dots, m$.
Let $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ take a meridian of L_i to ζ_i.
Let $F_i \subset D^{2n}$ be oriented compact smooth submanifolds transversal to each other, with $\partial F_i = F_i \cap \partial D^{2n} = L_i$.

In particular, $|\sigma_\zeta(L)| + \dim H_n(S^{2n-1} \smallsetminus L; \mathbb{C}_\mu)$
$\leq \dim H_n(F, L; \mathbb{Z}/p) + \dim H_{n-1}(F, L; \mathbb{Z}/p)$

Span inequalities

Let $L_1, \ldots, L_m \subset S^{2n-1}$ be smooth oriented transversal to each other submanifolds of codimension 2, $L = L_1 \cup \cdots \cup L_m$.

Let $\zeta_i \in \mathbb{C}$ be algebraic numbers with $|\zeta_i| = 1$, and f_i be irreducible integer polynomials with $f_i(\zeta_i) = 0$.
Suppose prime number p divides $f_i(1)$ for $i = 1, \ldots, m$.
Let $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ take a meridian of L_i to ζ_i.
Let $F_i \subset D^{2n}$ be oriented compact smooth submanifolds transversal to each other, with $\partial F_i = F_i \cap \partial D^{2n} = L_i$.

That is $$|\sigma_\zeta(L)| + n^0_\zeta(L) \leq \dim H_n(F, L; \mathbb{Z}/p) + \dim H_{n-1}(F, L; \mathbb{Z}/p)$$

Slice inequalities

Slice inequalities

Again, let $L_1, \dots, L_m \subset S^{2n-1}$ be smooth oriented transversal to each other submanifolds of codimension 2, $L = L_1 \cup \dots \cup L_m$.

Let $\zeta_i \in \mathbb{C}$ be algebraic numbers with $|\zeta_i| = 1$, and f_i be irreducible integer polynomials with $f_i(\zeta_i) = 0$.

Suppose prime number p divides $f_i(1)$ for $i = 1, \dots, m$.

Let $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ take a meridian of L_i to ζ_i.

Slice inequalities

Again, let $L_1, \dots, L_m \subset S^{2n-1}$ be smooth oriented transversal to each other submanifolds of codimension 2, $L = L_1 \cup \dots \cup L_m$.

Let $\zeta_i \in \mathbb{C}$ be algebraic numbers with $|\zeta_i| = 1$, and f_i be irreducible integer polynomials with $f_i(\zeta_i) = 0$.
Suppose prime number p divides $f_i(1)$ for $i = 1, \dots, m$.
Let $\mu : \pi_1(S^{2n-1} \setminus L) \to \mathbb{C}^\times$ take a meridian of L_i to ζ_i.

Let $\Lambda_i \subset S^{2n}$ be oriented closed smooth submanifolds transversal to each other and to S^{2n-1}, with $\partial\Lambda_i \cap S^{2n-1} = L_i$.

Slice inequalities

Again, let $L_1, \dots, L_m \subset S^{2n-1}$ be smooth oriented transversal to each other submanifolds of codimension 2,
$L = L_1 \cup \dots \cup L_m$.
Let $\zeta_i \in \mathbb{C}$ be algebraic numbers with $|\zeta_i| = 1$, and
f_i be irreducible integer polynomials with $f_i(\zeta_i) = 0$.
Suppose prime number p divides $f_i(1)$ for $i = 1, \dots, m$.
Let $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ take a meridian of L_i to ζ_i.

Let $\Lambda_i \subset S^{2n}$ be oriented closed smooth submanifolds transversal to each other and to S^{2n-1}, with $\partial\Lambda_i \cap S^{2n-1} = L_i$.
Put $\Lambda = \cup_i \Lambda_i$. Extend $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ to
$\mu : \pi_1(S^{2n} \smallsetminus \Lambda) \to \mathbb{C}^\times$.

Slice inequalities

Again, let $L_1, \dots, L_m \subset S^{2n-1}$ be smooth oriented transversal to each other submanifolds of codimension 2, $L = L_1 \cup \dots \cup L_m$.

Let $\zeta_i \in \mathbb{C}$ be algebraic numbers with $|\zeta_i| = 1$, and f_i be irreducible integer polynomials with $f_i(\zeta_i) = 0$. Suppose prime number p divides $f_i(1)$ for $i = 1, \dots, m$. Let $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ take a meridian of L_i to ζ_i.

Let $\Lambda_i \subset S^{2n}$ be oriented closed smooth submanifolds transversal to each other and to S^{2n-1}, with $\partial\Lambda_i \cap S^{2n-1} = L_i$. Put $\Lambda = \cup_i \Lambda_i$. Extend $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ to $\mu : \pi_1(S^{2n} \smallsetminus \Lambda) \to \mathbb{C}^\times$.

Theorem. $|\sigma_\zeta(L)| \leq \frac{1}{2} \dim H_{n-1}(\Lambda; \mathbb{Z}/p)$

Slice inequalities

Again, let $L_1, \ldots, L_m \subset S^{2n-1}$ be smooth oriented transversal to each other submanifolds of codimension 2, $L = L_1 \cup \cdots \cup L_m$.

Let $\zeta_i \in \mathbb{C}$ be algebraic numbers with $|\zeta_i| = 1$, and f_i be irreducible integer polynomials with $f_i(\zeta_i) = 0$.
Suppose prime number p divides $f_i(1)$ for $i = 1, \ldots, m$.
Let $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ take a meridian of L_i to ζ_i.

Let $\Lambda_i \subset S^{2n}$ be oriented closed smooth submanifolds transversal to each other and to S^{2n-1}, with $\partial\Lambda_i \cap S^{2n-1} = L_i$.
Put $\Lambda = \cup_i \Lambda_i$. Extend $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ to $\mu : \pi_1(S^{2n} \smallsetminus \Lambda) \to \mathbb{C}^\times$.

Theorem. $|\sigma_\zeta(L)| \le \frac{1}{2} \dim H_{n-1}(\Lambda; \mathbb{Z}/p)$

$$|\sigma_\zeta(L)| + n^0_\zeta(L) \le \frac{1}{2} \dim H_{n-1}(\Lambda; \mathbb{Z}/p) + \dim H_{n-2}(\Lambda \smallsetminus L; \mathbb{Z}/p)$$

Slice inequalities

Again, let $L_1, \dots, L_m \subset S^{2n-1}$ be smooth oriented transversal to each other submanifolds of codimension 2, $L = L_1 \cup \dots \cup L_m$.

Let $\zeta_i \in \mathbb{C}$ be algebraic numbers with $|\zeta_i| = 1$, and f_i be irreducible integer polynomials with $f_i(\zeta_i) = 0$. Suppose prime number p divides $f_i(1)$ for $i = 1, \dots, m$. Let $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ take a meridian of L_i to ζ_i.

Let $\Lambda_i \subset S^{2n}$ be oriented closed smooth submanifolds transversal to each other and to S^{2n-1}, with $\partial\Lambda_i \cap S^{2n-1} = L_i$. Put $\Lambda = \cup_i \Lambda_i$. Extend $\mu : \pi_1(S^{2n-1} \smallsetminus L) \to \mathbb{C}^\times$ to $\mu : \pi_1(S^{2n} \smallsetminus \Lambda) \to \mathbb{C}^\times$.

Theorem. $|\sigma_\zeta(L)| \le \frac{1}{2} \dim H_{n-1}(\Lambda; \mathbb{Z}/p)$

$$|\sigma_\zeta(L)| + n^r_\zeta(L) \le \frac{1}{2} \sum_{s=-2r}^{2r} (-1)^s \dim H_{n-1+s}(\Lambda; \mathbb{Z}/p) + \sum_{s=0}^{2r} (-1)^s \dim H_{n-2-s}(\Lambda \smallsetminus L; \mathbb{Z}/p)$$