Complex tropical geometry

About strange objects which lie behind the basic objects of the tropical geometry, just between tropical and classical algebraic geometries

Oleg Viro

November 30, 2009

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Any abelian group is an mv-group.

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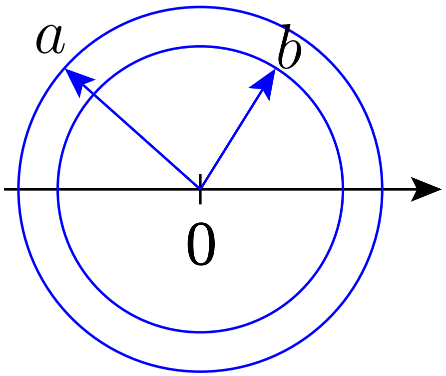

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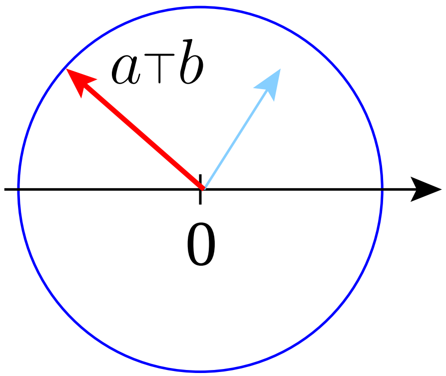

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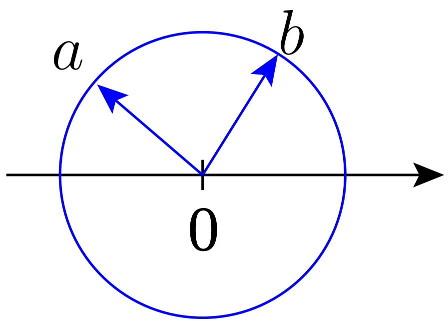

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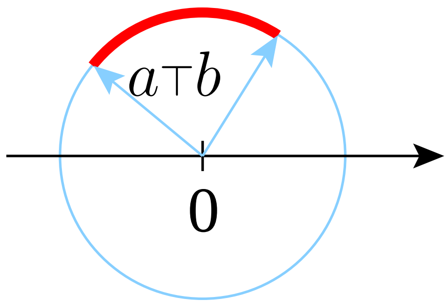

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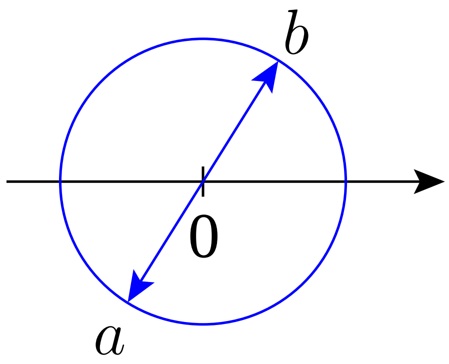

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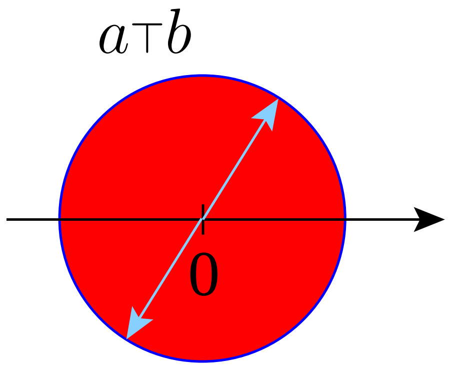

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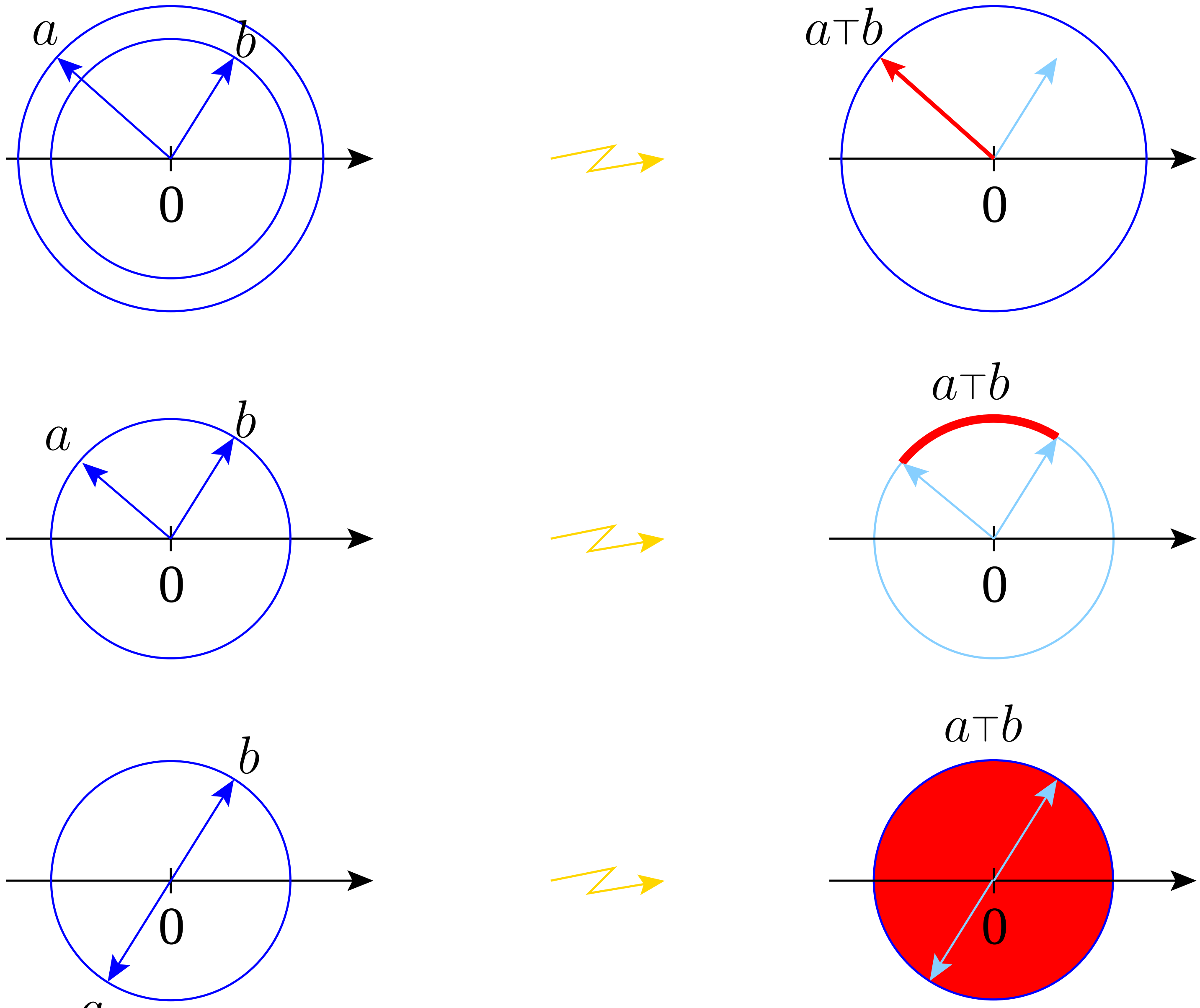

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Recall that the definition of multivalued binary operation prohibits $g(a, b)$ to be empty.

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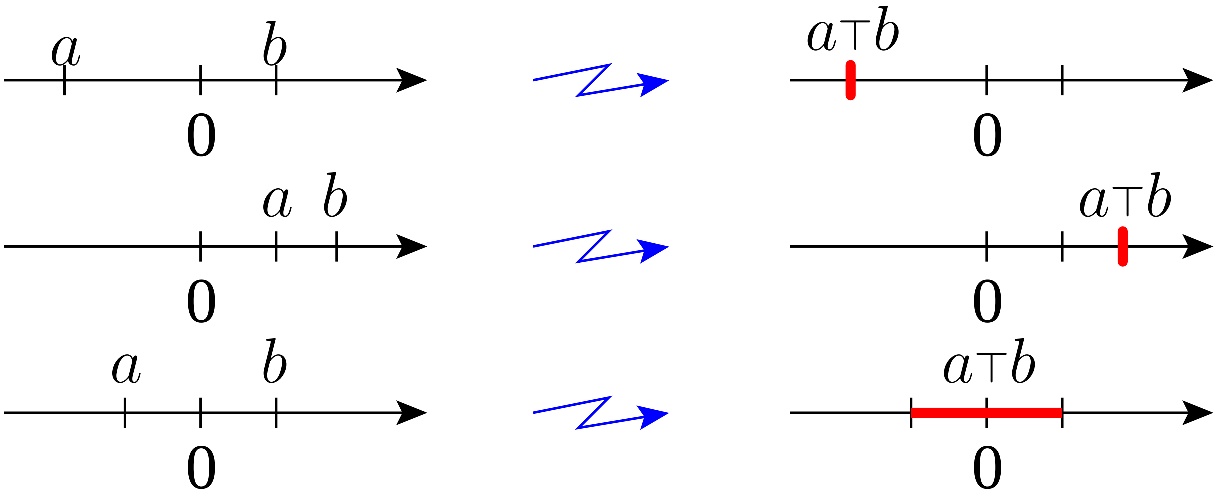

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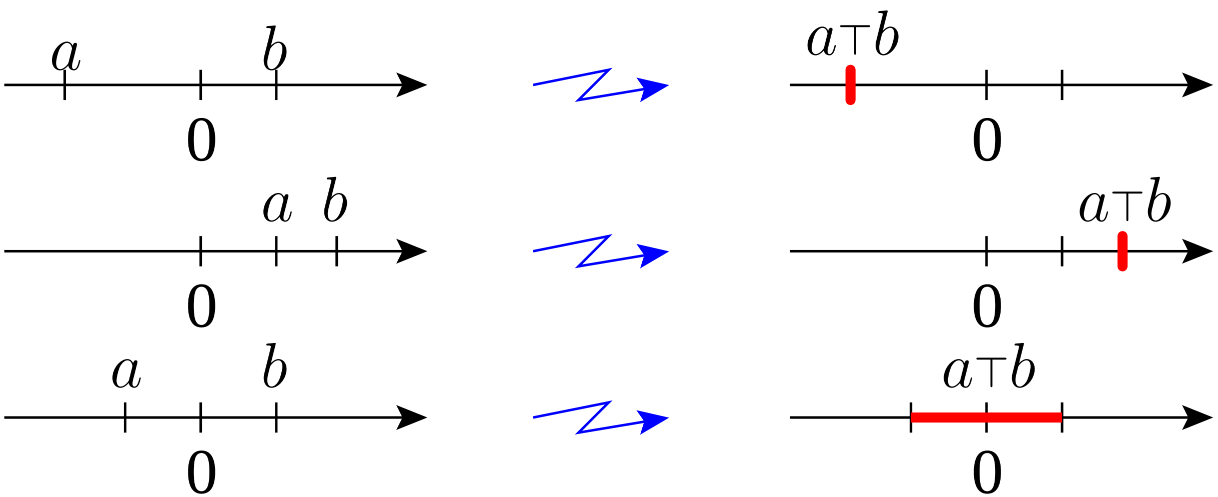

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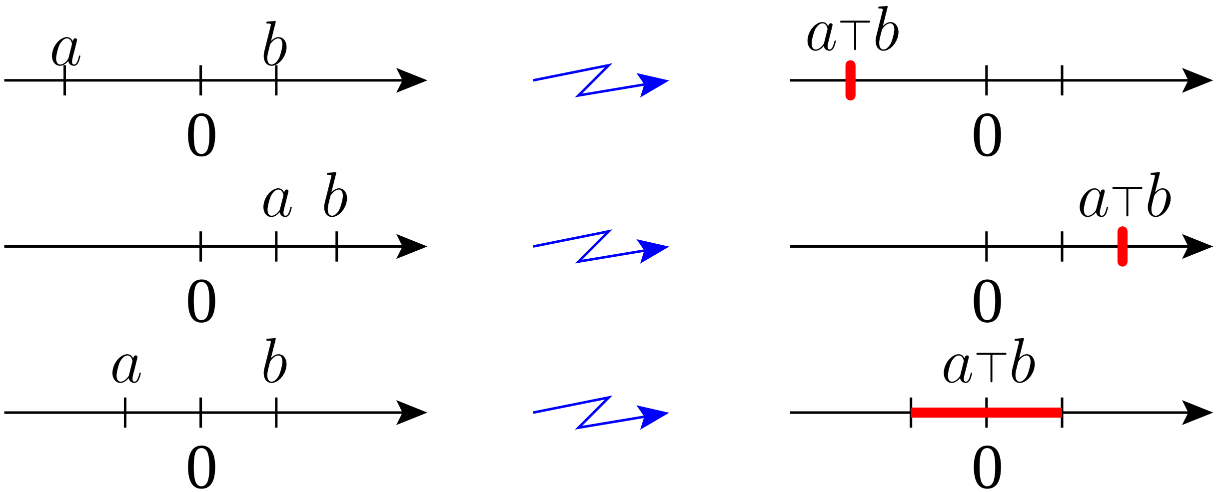

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then $(Y, \top_Y)$ is an mv-group (mv-subgroup of X) and $Y \hookrightarrow X$ is a homomorphism.

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A natural map in the opposite direction: $\mathbb{R}_{\top} \to \mathbb{R}_{\geq 0, \max} : x \mapsto |x|$ is not a homomorphism.

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Substitution $q = e^h$ transforms $\sum_n a_n q^{r_n}$ to $\sum_n a_n e^{hr_n}$ and provides an interpretation as functions $\mathbb{C} \to \mathbb{C}$.

Let $a_M q^{r_M}$ be the summand of $\sum_n a_n q^{r_n}$ with greatest r_n .

Define a map $\mathbb{C}[\mathbb{R}] \to \mathbb{C}$ which takes $\sum_n a_n q^{r_n}$ to $\dfrac{a_M}{|a_M|} e^{r_M}$.

This is a homomorphism $f : \mathbb{C}[\mathbb{R}] \to \mathbb{C}_{\top,\times}$:
$f(a + b) \in f(a) \top f(b)$ and $f(ab) = f(a)f(b)$.

Dequantiztaion

Deformation of $\mathbb{C}$

For $h > 0$ consider a map $S_h\colon \mathbb{C} \to \mathbb{C}$

$$z \longmapsto \begin{cases} |z|^{\frac{1}{h}} \frac{z}{|z|} = |z|^{\frac{1-h}{h}} z, & \text{if } z \neq 0; \\ 0, & \text{if } z = 0. \end{cases}$$

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$\mathbb{C}_h = \mathbb{C}_{+_h,\times}$ is a copy of $\mathbb{C}$ and $S_h : \mathbb{C}_h \to \mathbb{C}$ is an isomorphism.

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If $z + w = 0$, then $\lim_{h\to 0}(z +_h w) = 0$.
Denote $\lim_{h\to 0}(z +_h w)$ by $z +_0 w$.

A look of the limit

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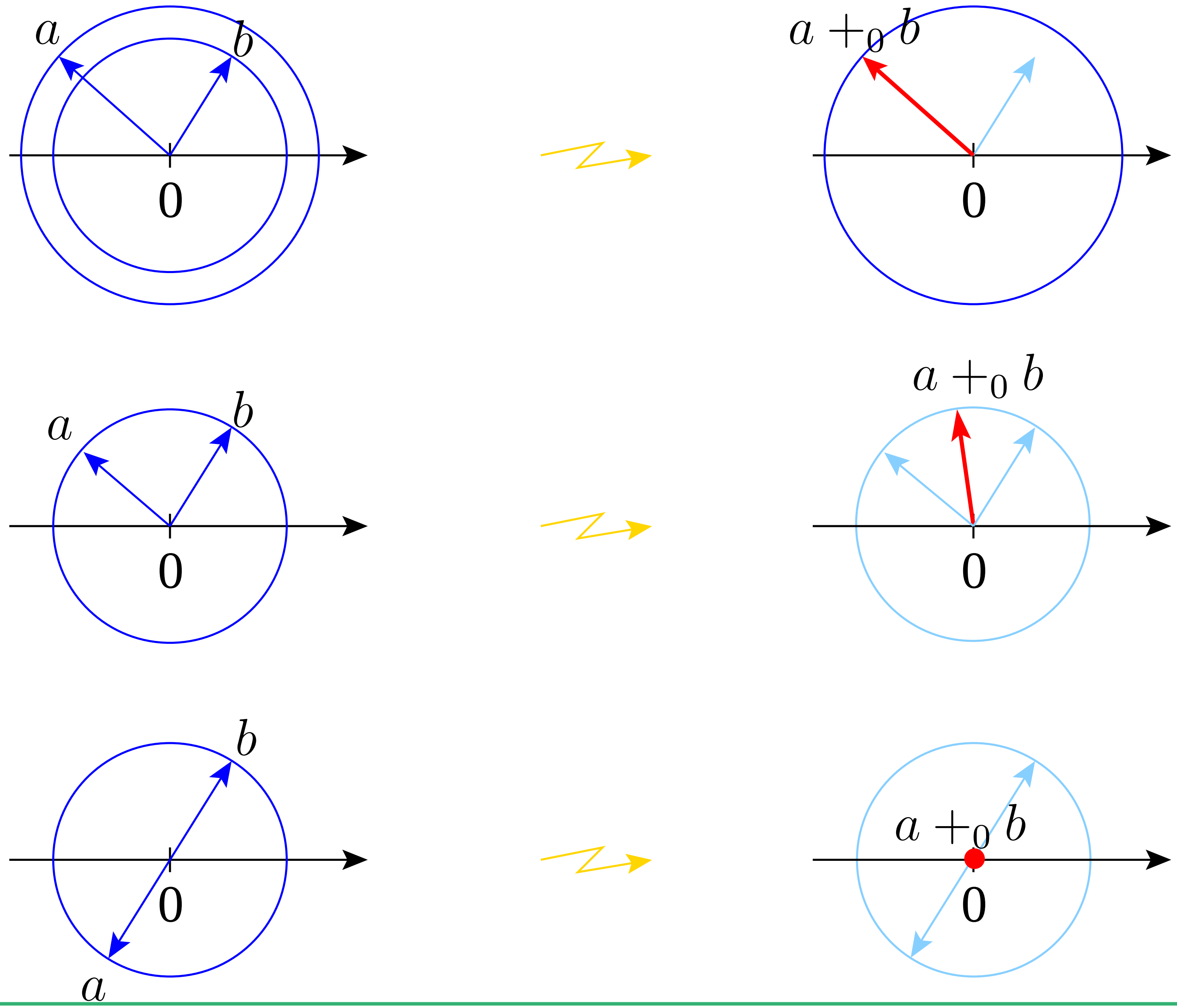

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- commutative,
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There is one that
fixes all the defects,
but gives a **multivalued** $\top$!

Upper Vietoris topology

Let X be a topological space.

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For example $A = X$!

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If X is normal, then the set of all such closed A is a filter,
but is not closed against intersection.

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If X is first countable and regular, then $\mathrm{LIM}_{h\to 0} F_h$
is the set of limits of $x_h \in F_h$ as $h \to 0$.

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If $F_h \in X \times Y$ are graphs of univalent continuous maps $X \to Y$,
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If the images of points are compact, and the graph is closed,
then the multivalent map is upper semi-continuous.
If the images of points are connected and the map is upper semi-continuous, then the image of a connected set is connected.
If the images of points are compact and the map is upper semi-continuous, then the image of a compact set is compact.

Topology of tropical addition

Let $\Gamma_h \subset \mathbb{C}^3$ be a graph of $+_h$ for $h > 0$:

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Hence, $\top$ is upper semi-continuous and

maps a connected set to a connected set

and a compact set to a compact set.

Equations and varieties

Good and bad polynomials

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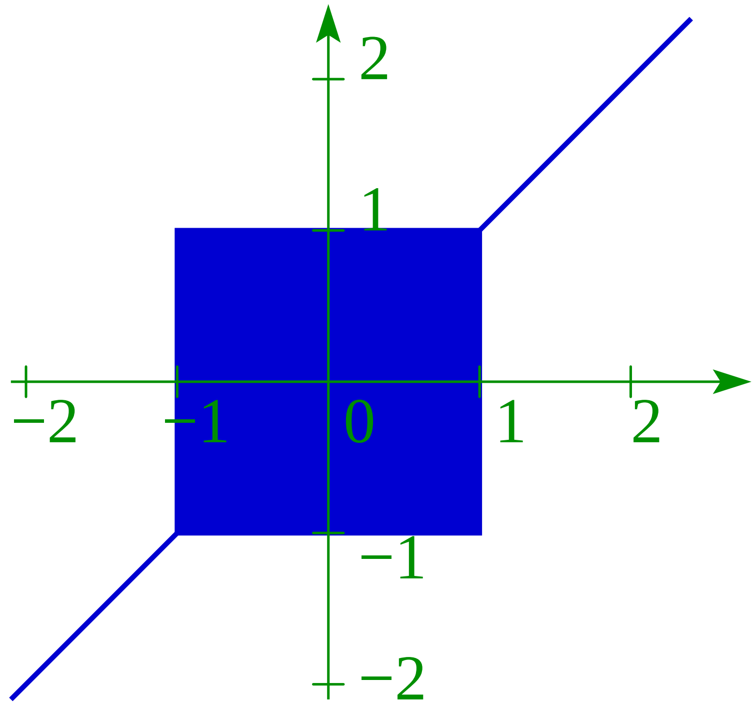

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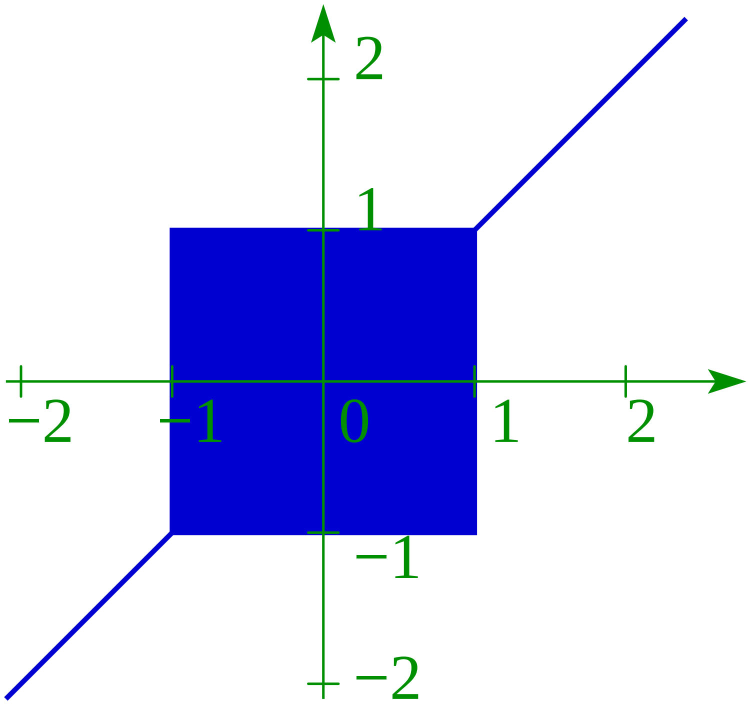

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A polynomial is said to be pure if it has no two monomials with the same exponents.

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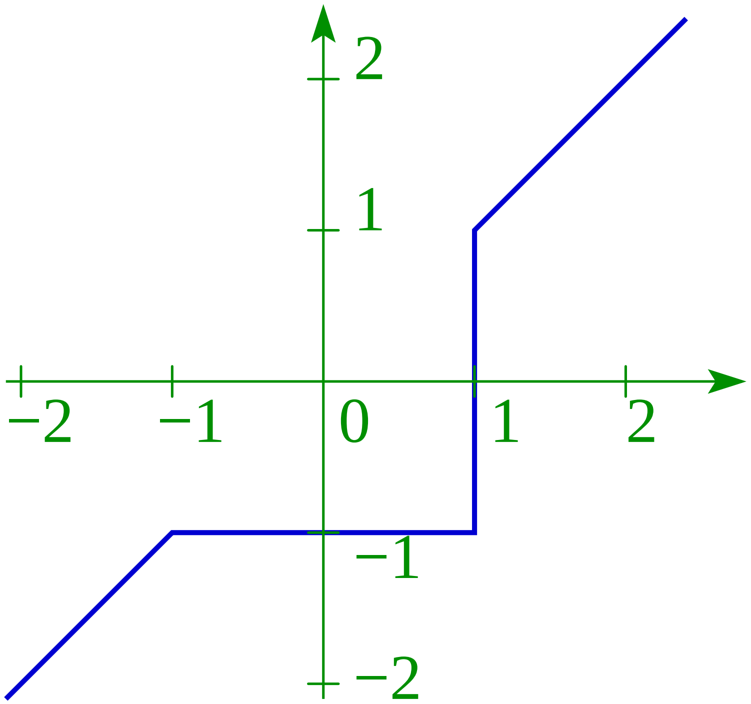

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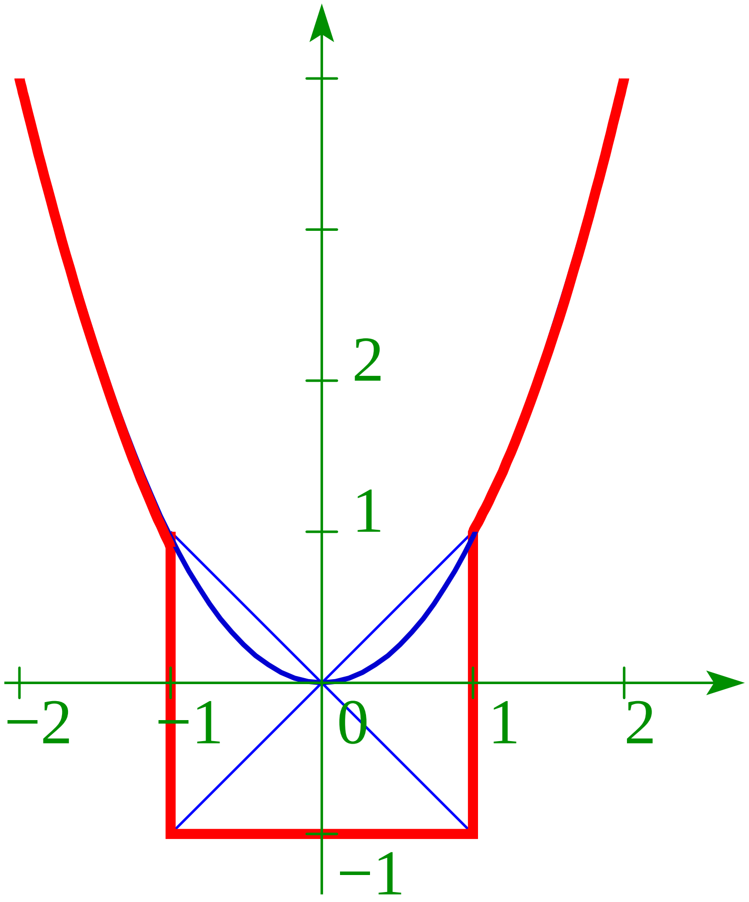

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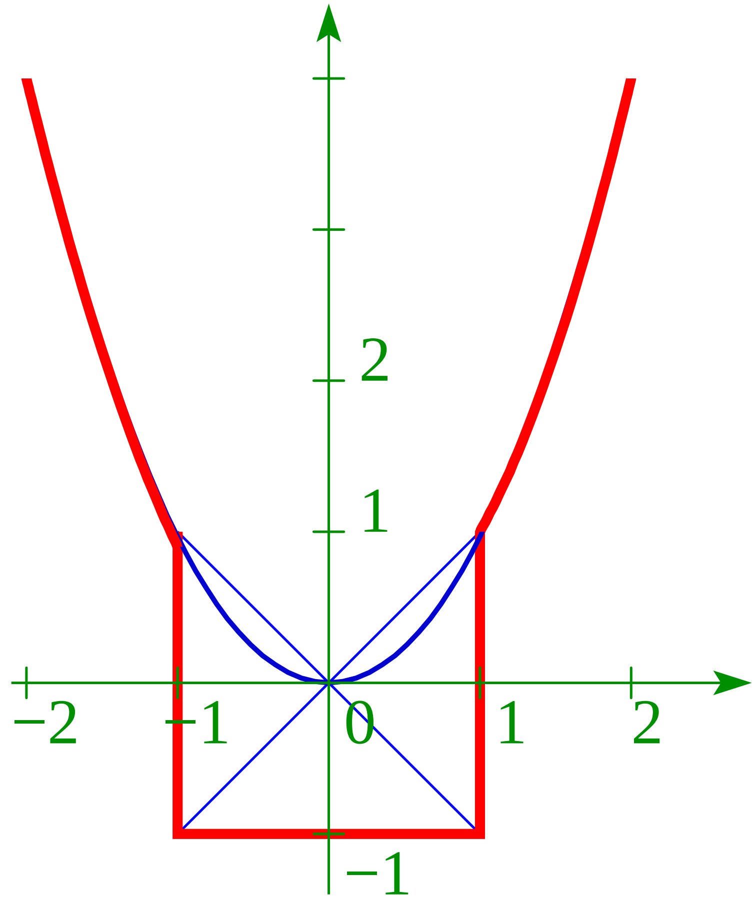

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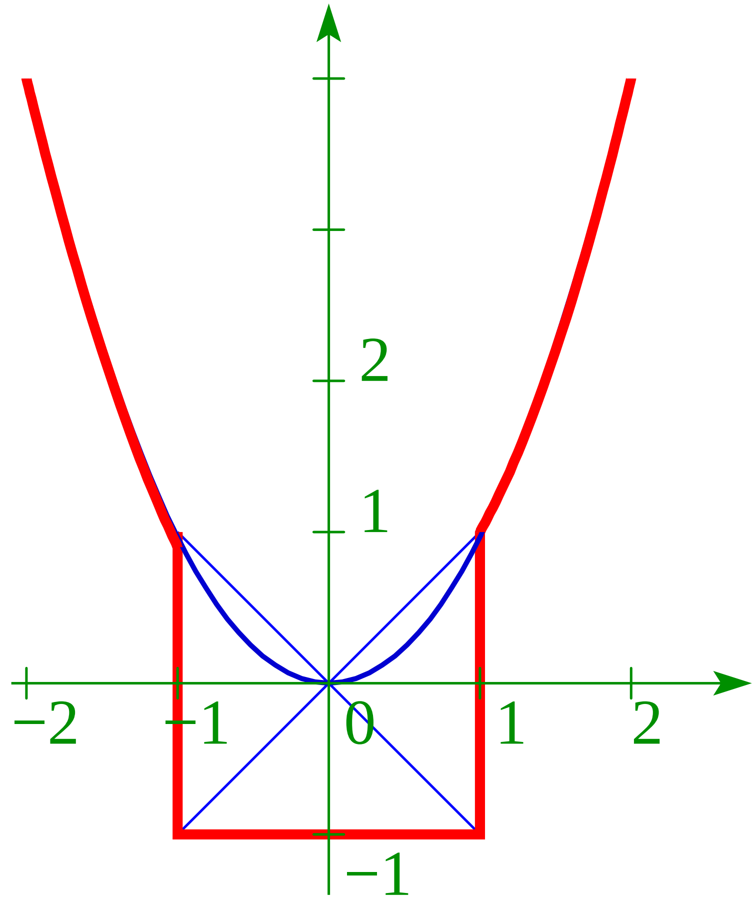

The graph of a polynomial is connected.

Because a polynomial is upper semi-continuous and has connected values.

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Then only those with the greatest one matter!

Amoebas: relation to tropics

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$(\mathbb{C} \setminus 0)^n$ is convenient to consider fibred over $\mathbb{R}^n$ via the map
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Let

$$p(z_1, \dots, z_n) = \underset{k=(k_1,\dots k_n)\in I}{\top} a_k z_1^{k_1} \dots z_n^{k_n}$$

be a pure $\top$-polynomial.

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be a pure $\top$-polynomial. Let

$$q(x_1, \ldots, x_n) = \max\{\log|a_k| + k_1 \log|x_1| + \cdots + k_n \log|x_n| \mid k \in I\}$$

be its tropical version (in a sense, $\mathrm{Log}_!(p)$?).

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Let

$$p(z_1, \dots, z_n) = \top_{k=(k_1,\dots k_n)\in I} a_k z_1^{k_1} \dots z_n^{k_n}$$

be a pure $\top$-polynomial. Let

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