Patchworking of algebraic varieties and tropical geometry

Oleg Viro

February 8, 2008

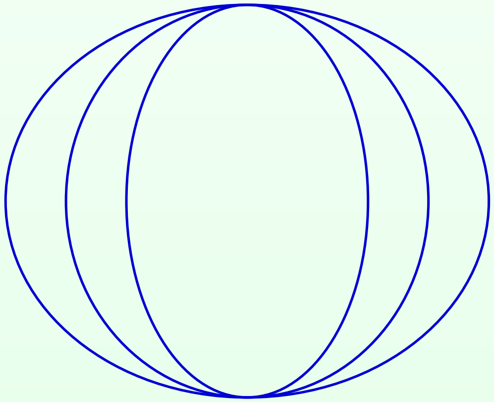
Patchwork

Construction of sextics

Here is how the patchwork works:

Construction of sextics

53 out of 56 topological types of non-singular sextics can be realized by permutation of the union of 3 ellipses tangent to each other at 2 points.



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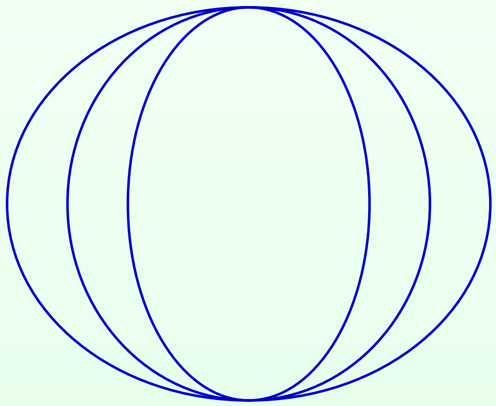

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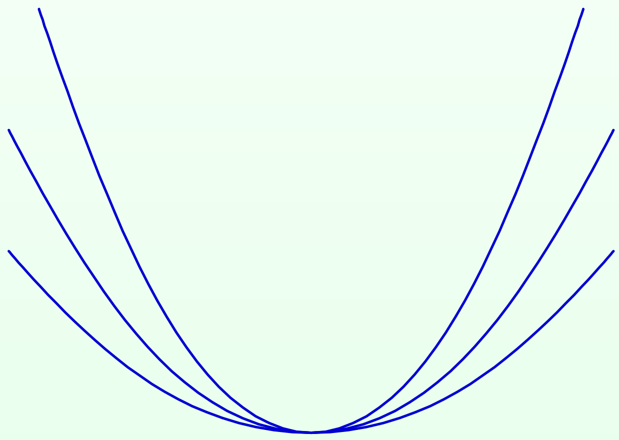

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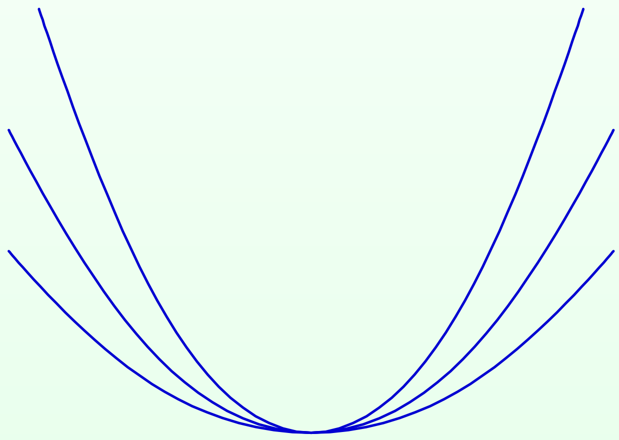

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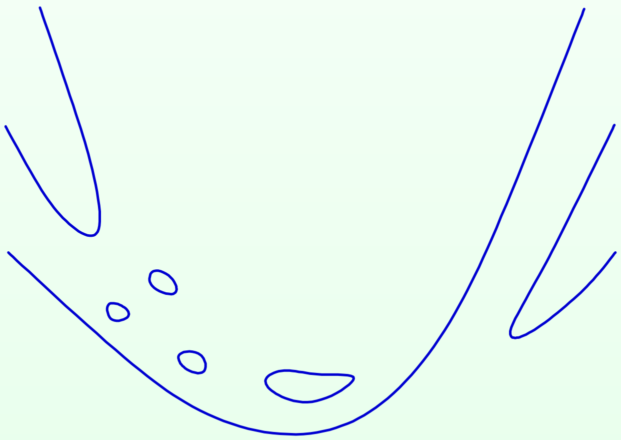

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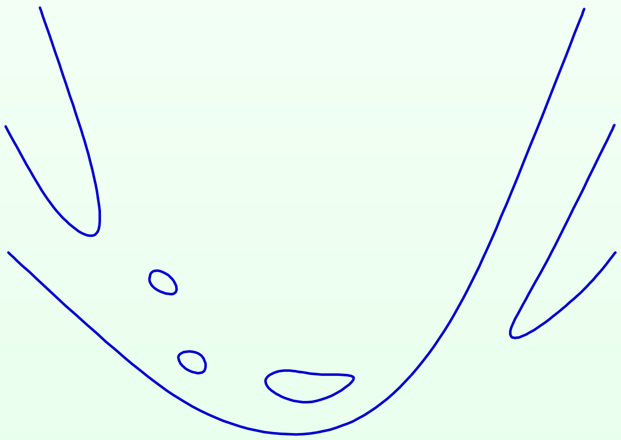

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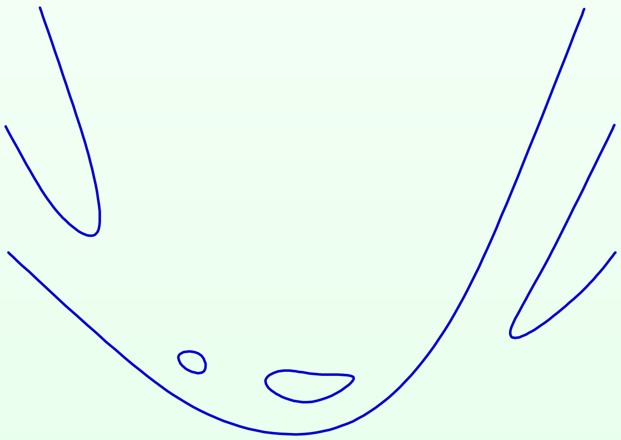

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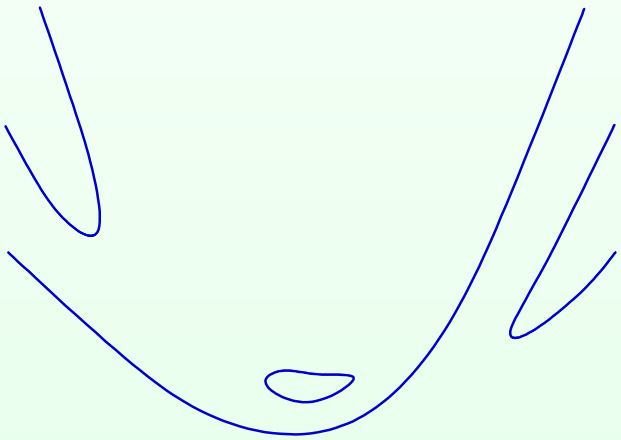

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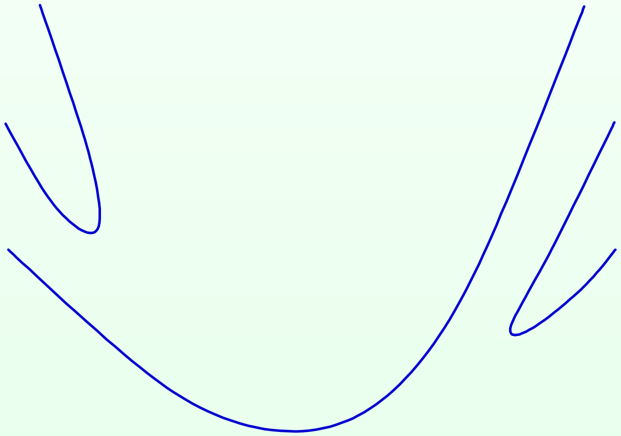

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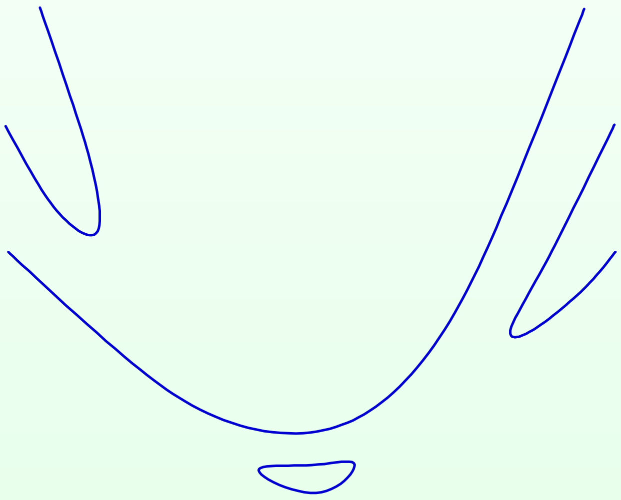

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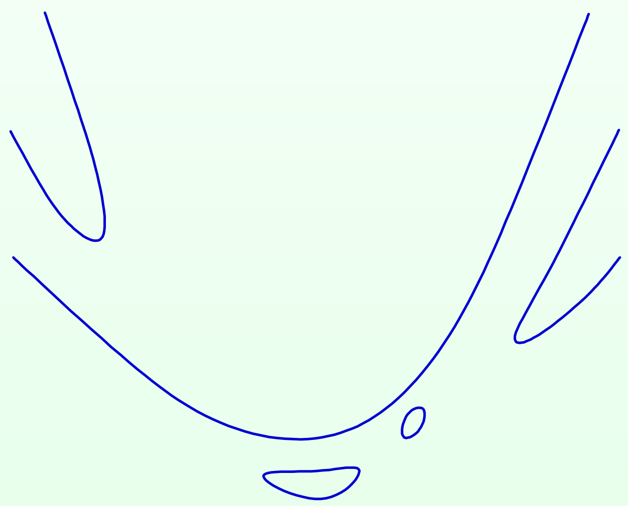

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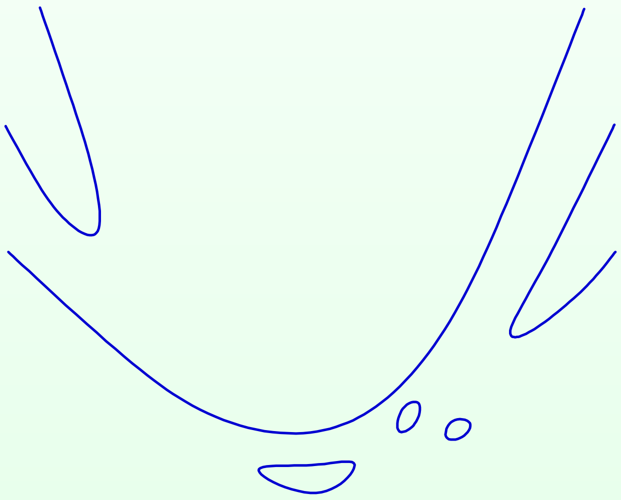

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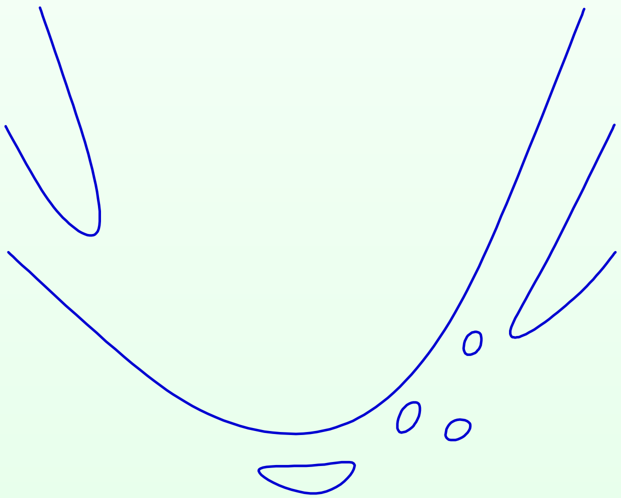

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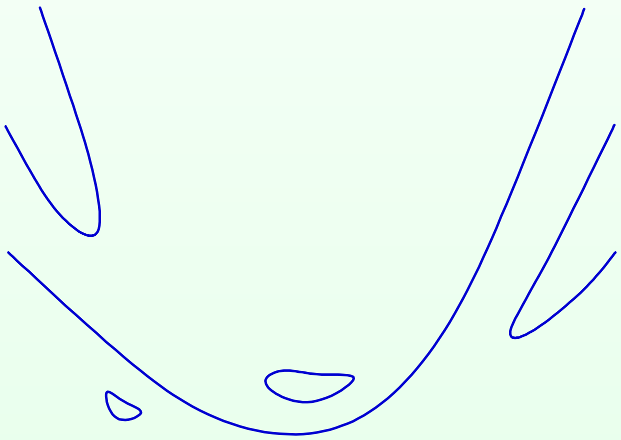

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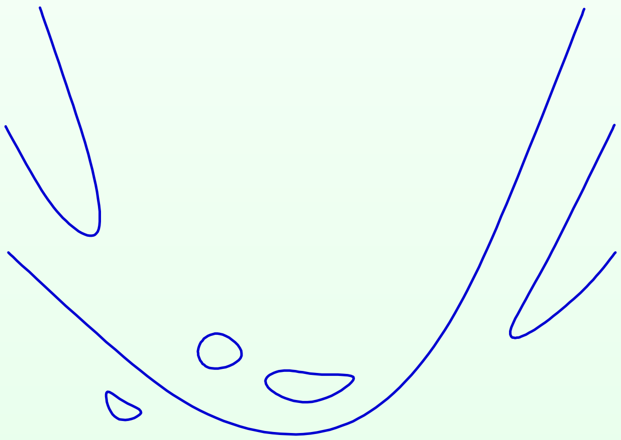

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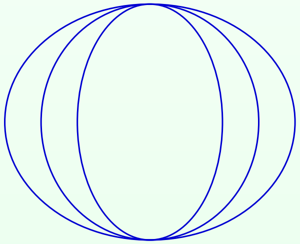

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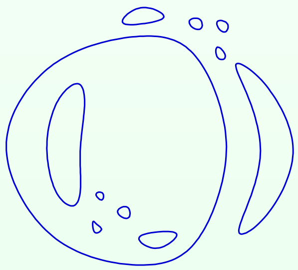

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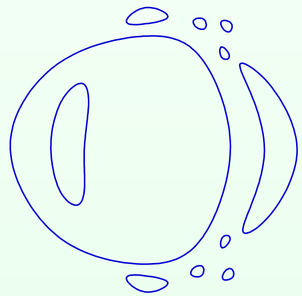

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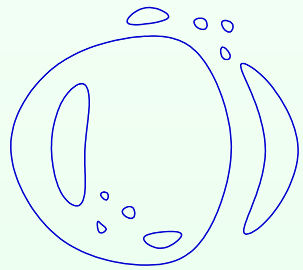

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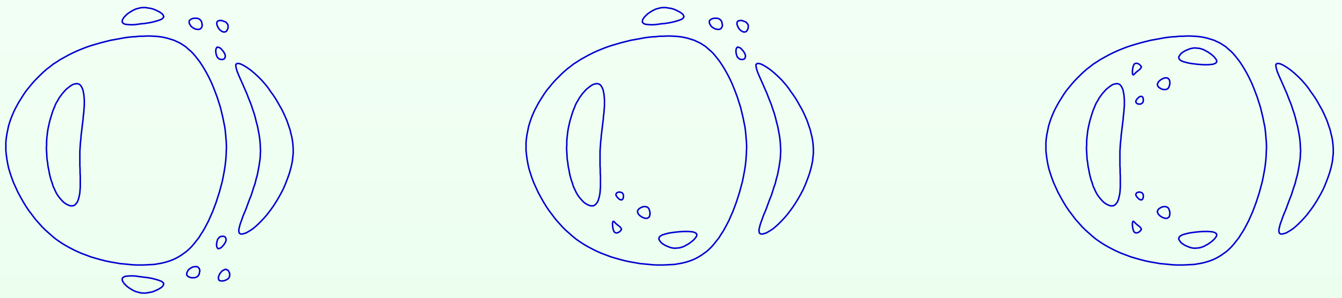

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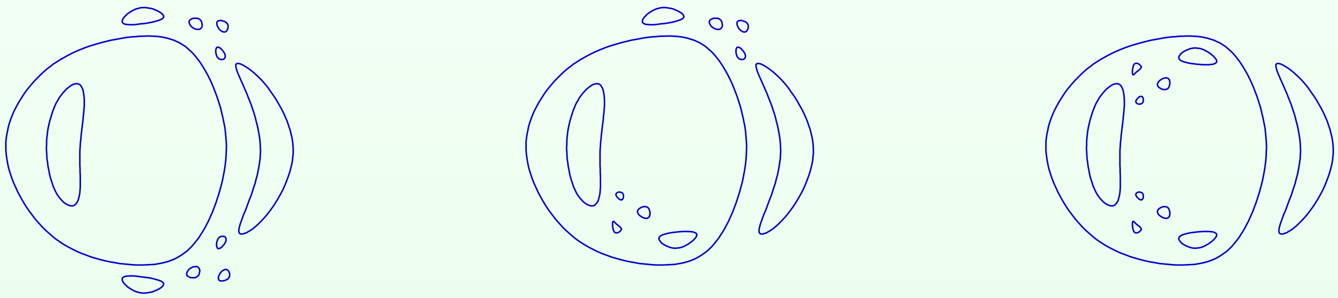

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Similarly non-singular curves of degree 7 of all topological types unrealized by 1979 are obtained from four curves with two singular points of the same kind.

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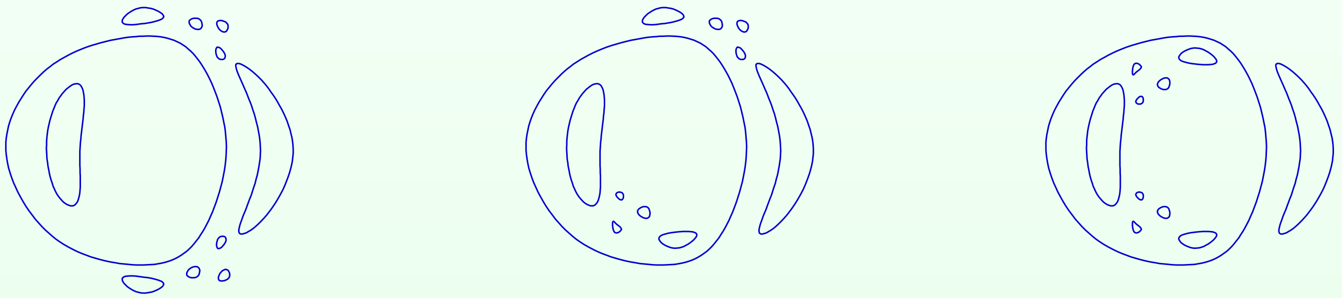

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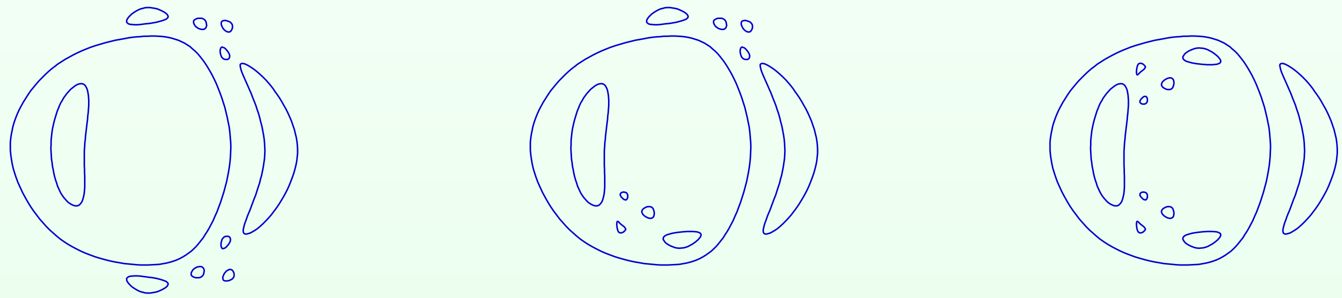

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What are the equations of the curves?

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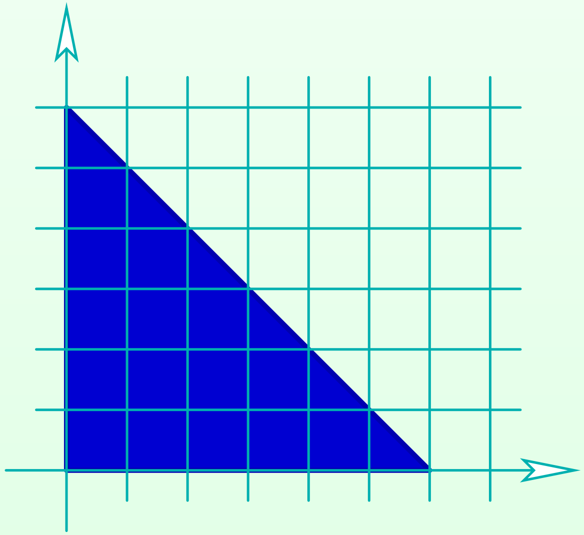

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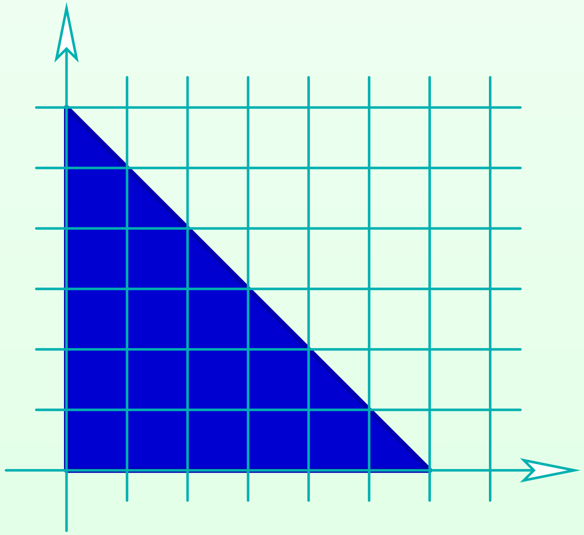

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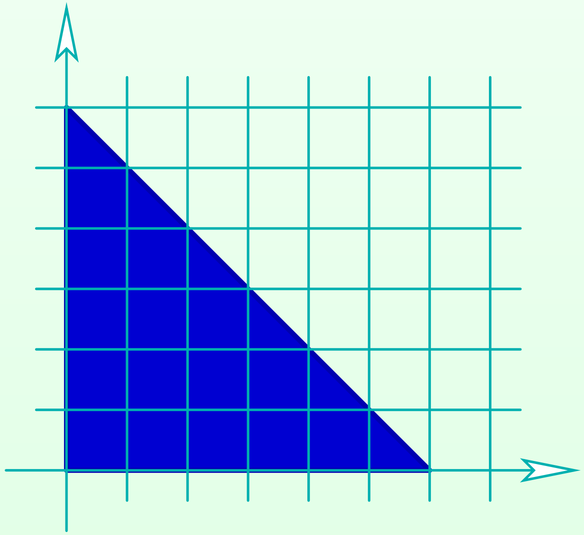

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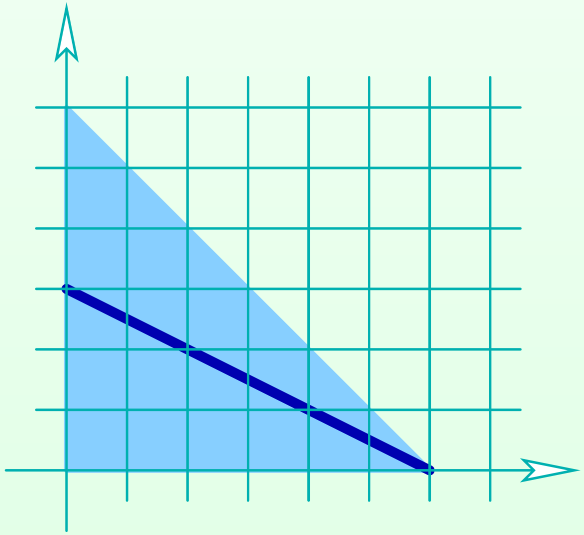
Then the Newton polygon is $[(6,0),(0,3)]$.



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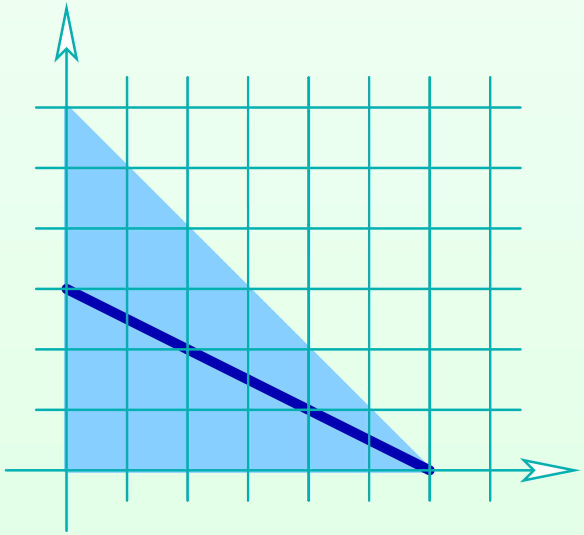
To perturb, we fill the two missing triangles with equations of curves we want to insert instead of neighborhoods of the singular points.



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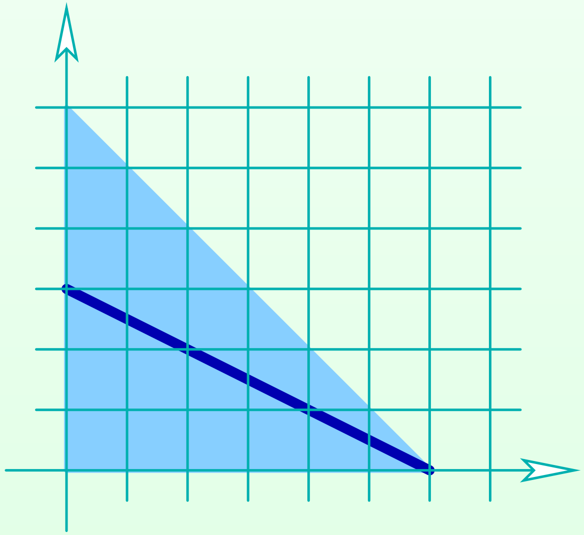
Introduce a small parameter $t > 0$ to keep the new fragments of the polynomial in peace with each other.



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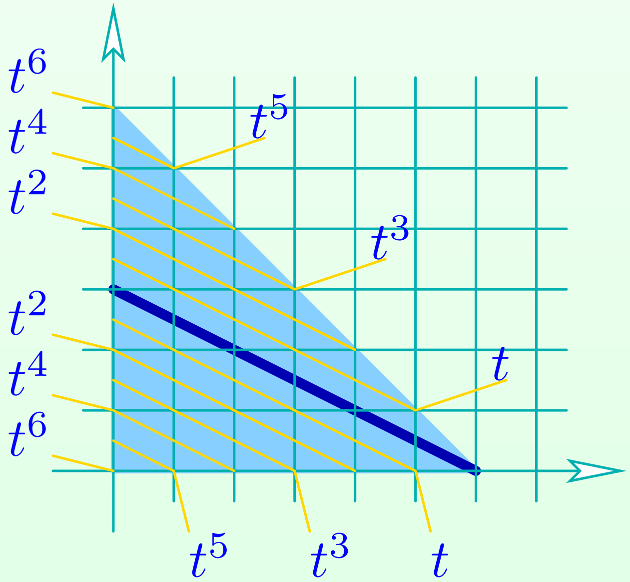
For sufficiently small t, the fragments defined by small terms are small, separated and do not spoil each other.



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Similarly, any binomial equation $y^l = ax^k$ defines line $lv = ku + b$.

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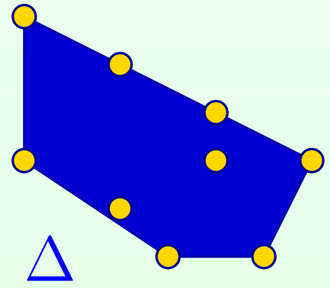

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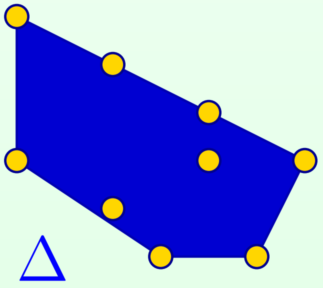

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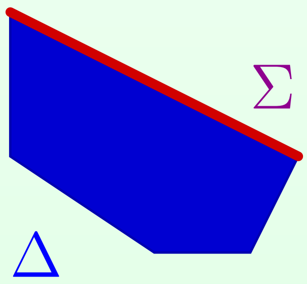

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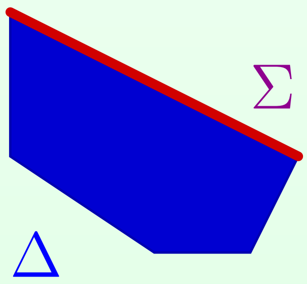

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Let a be a real polynomial in x, y,
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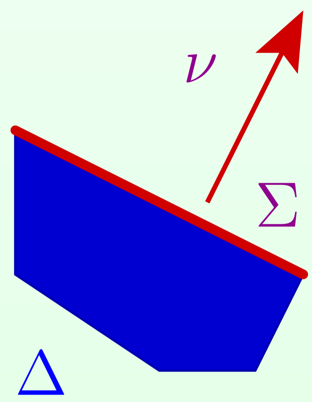
and Δ the Newton polygon of a. Let Σ be a side of Δ,
$\nu = (m, n)$ be an integer vector orthogonal to Σ.



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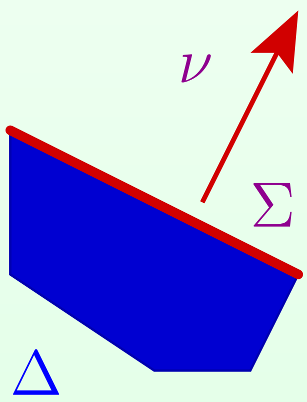
Go in the direction of ν



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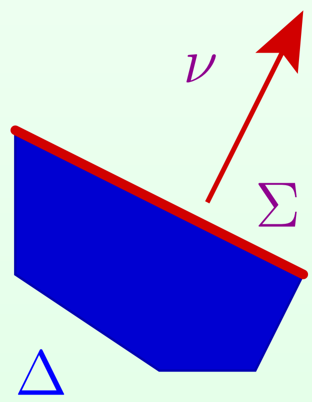
Go in the direction of ν
$(u, v) \mapsto (mt + u, nt + v)$



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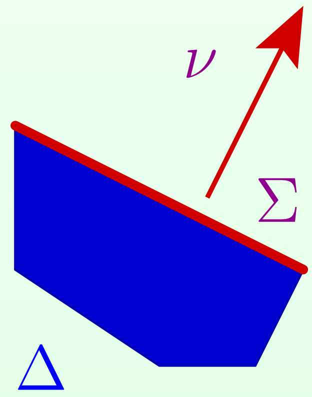

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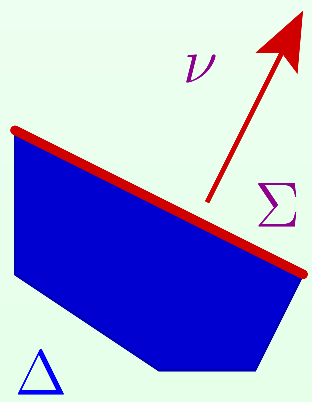
$$(u, v) \mapsto (mt + u, nt + v)$$

$$a(e^u, e^v) = 0 \quad \mapsto \quad a(e^{mt+u}, e^{nt+v}) = 0$$



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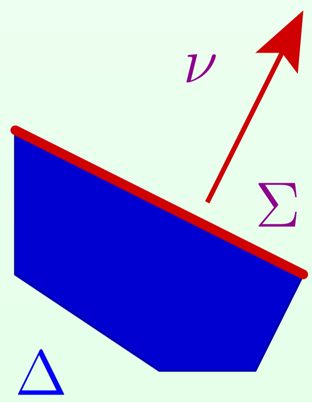
Go in the direction of ν looking at V.

$$(u, v) \mapsto (mt + u, nt + v)$$

$$\sum a_{k,l} e^{ku+lv} = 0 \mapsto \sum a_{k,l} e^{k(mt+u)+l(nt+v)} = 0$$



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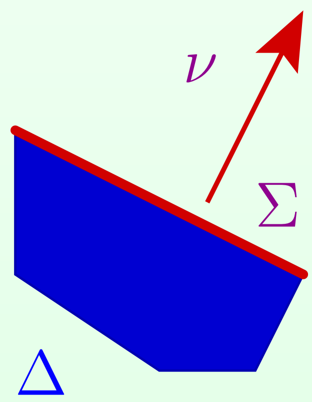
Go in the direction of ν looking at V.

$$(u, v) \mapsto (mt + u, nt + v)$$

$$\sum a_{k,l} e^{ku+lv} = 0 \;\mapsto\; \sum a_{k,l} e^{(km+ln)t} e^{ku+lv} = 0$$



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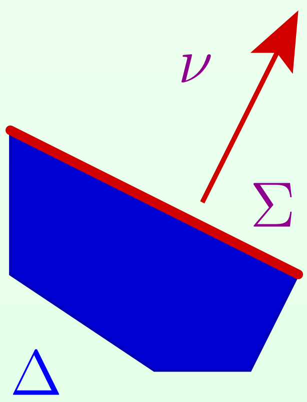

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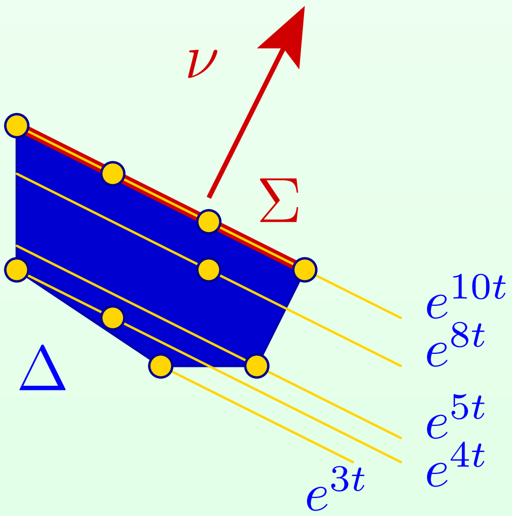

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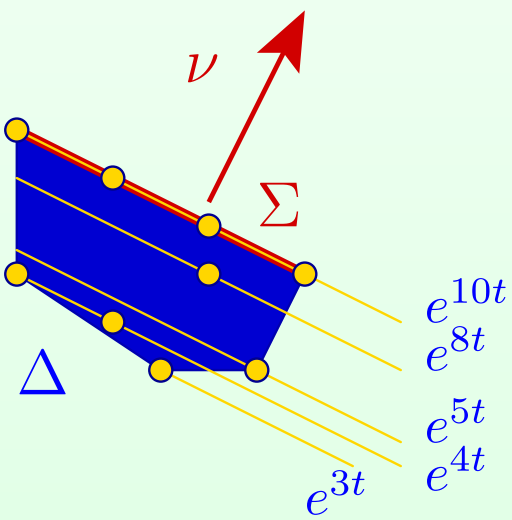
$\sum a_{k,l} e^{ku+lv} = 0 \mapsto \sum (a_{k,l} e^{(km+ln)t}) e^{ku+lv} = 0$



All the coefficients tend to ∞.

Logarithmic asymptotes

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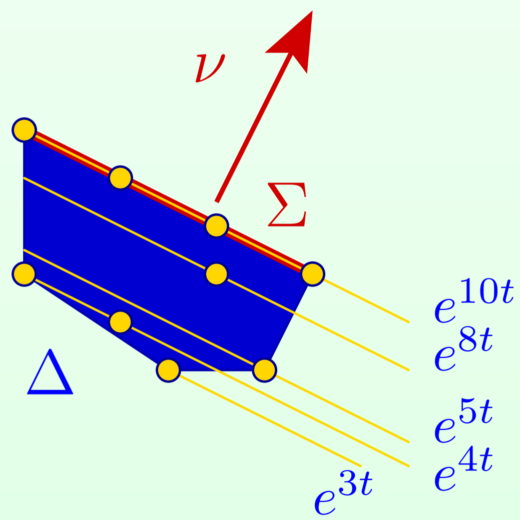

Calibrate!

Logarithmic asymptotes

Let a be a real polynomial in x, y,
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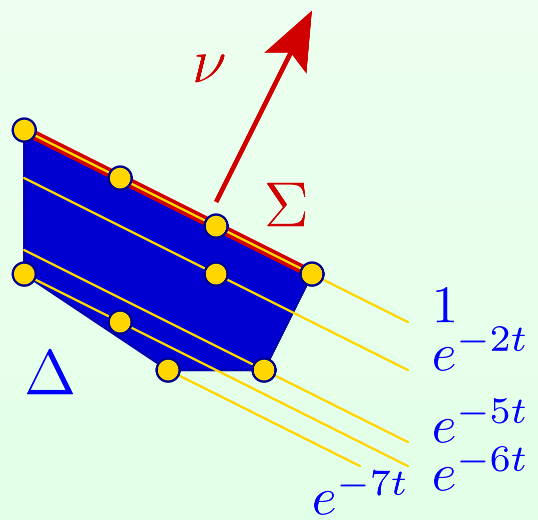

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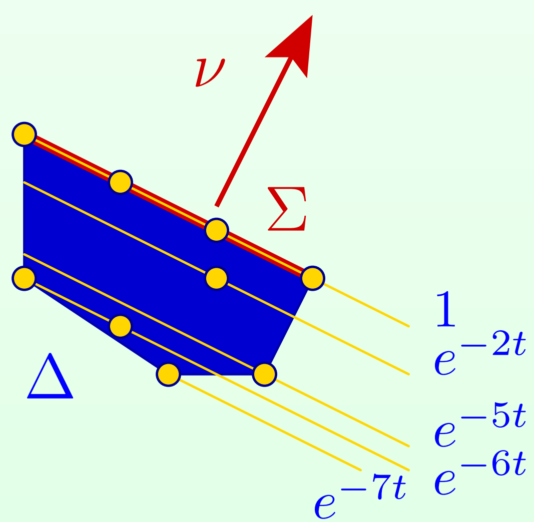
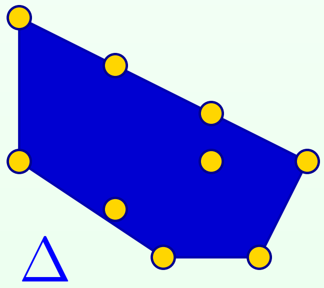


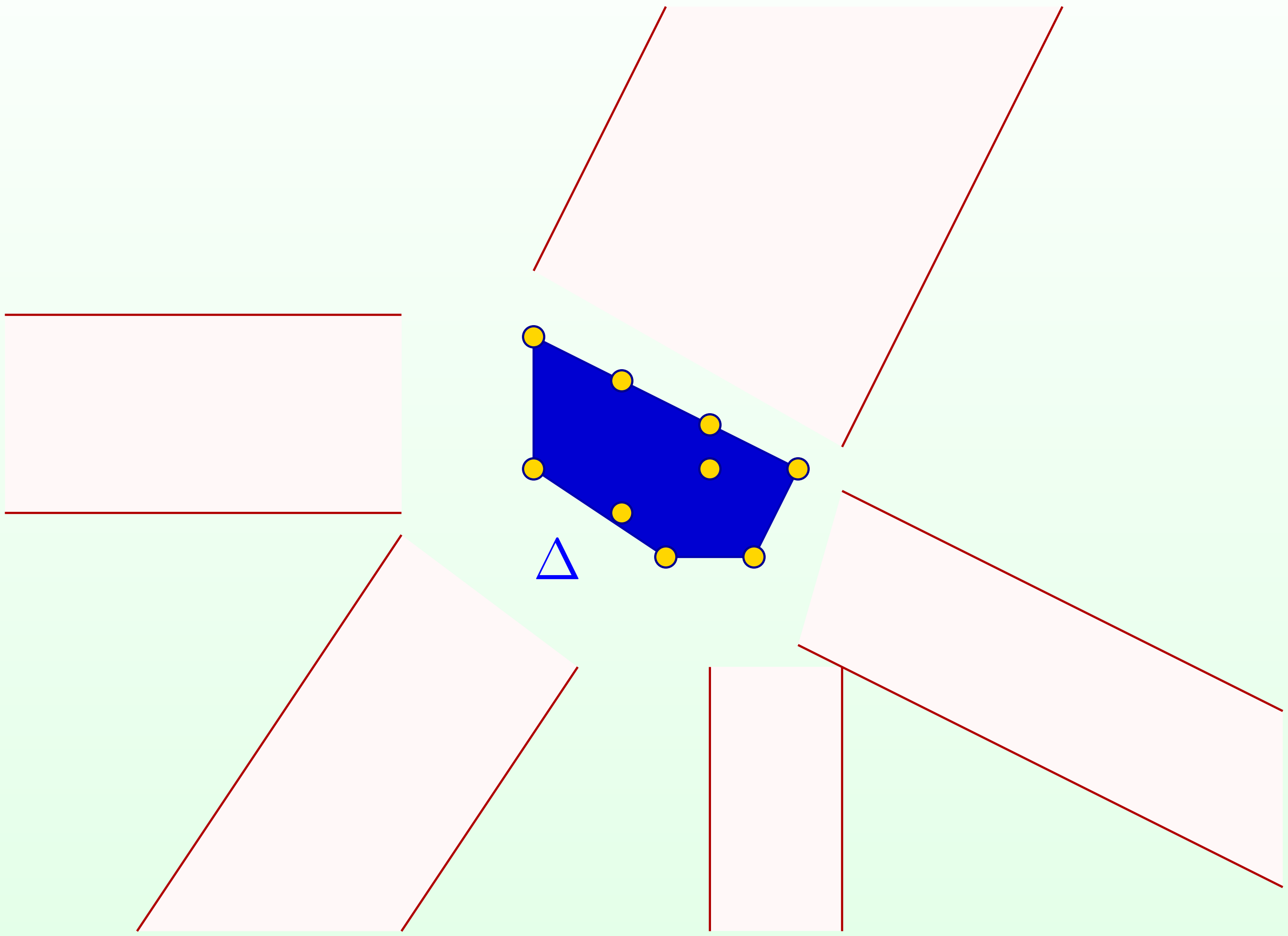
$a(e^{mt+u}, e^{nt+v})$ tends to
$a^\Sigma(u, v) = \sum_{(k,l)\in\Sigma} a_{kl} e^{ku+lv}$
as $t \to \infty$.

Picture of logarithmic asymptotes



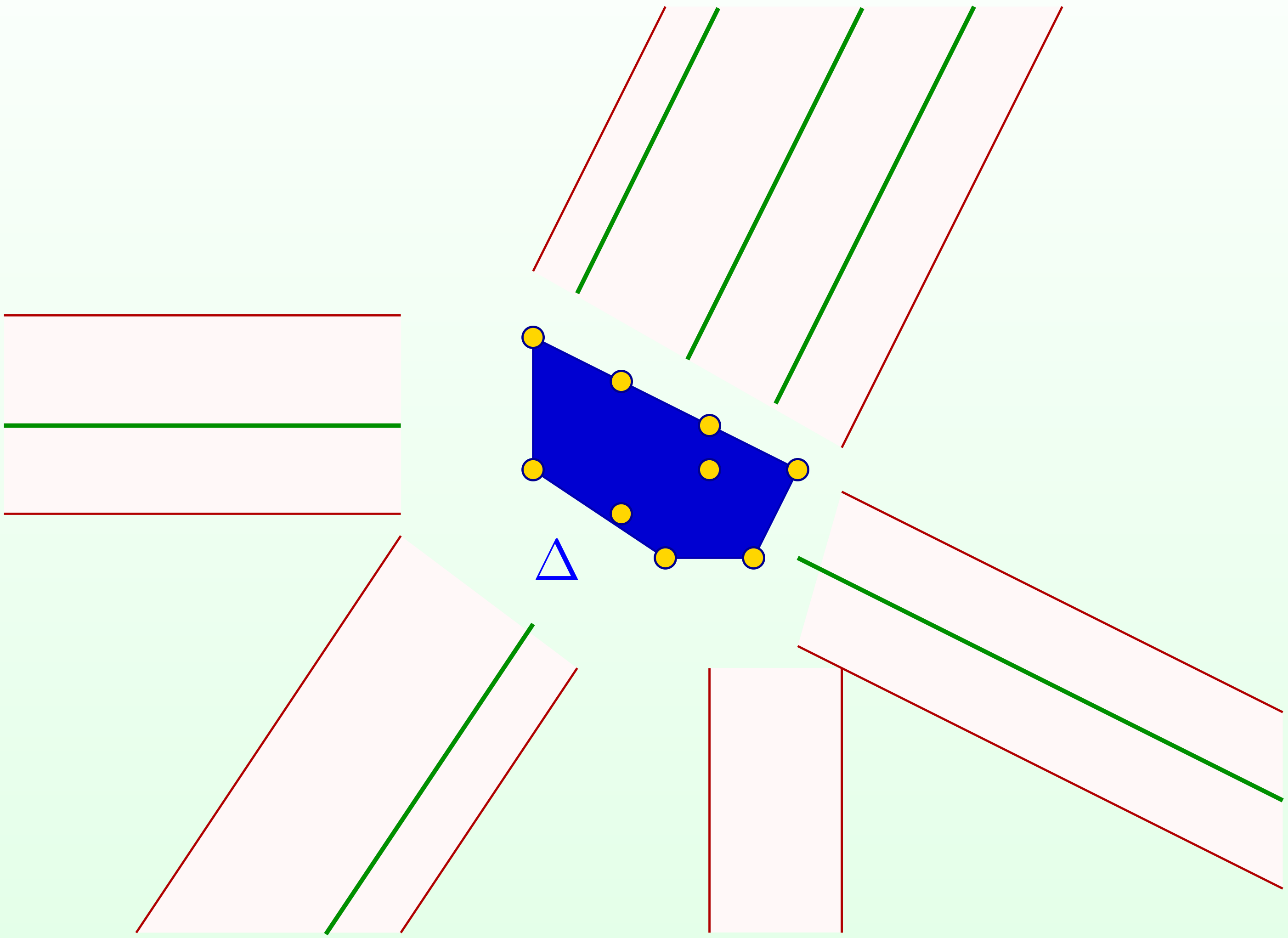
Newton polygon.

Picture of logarithmic asymptotes



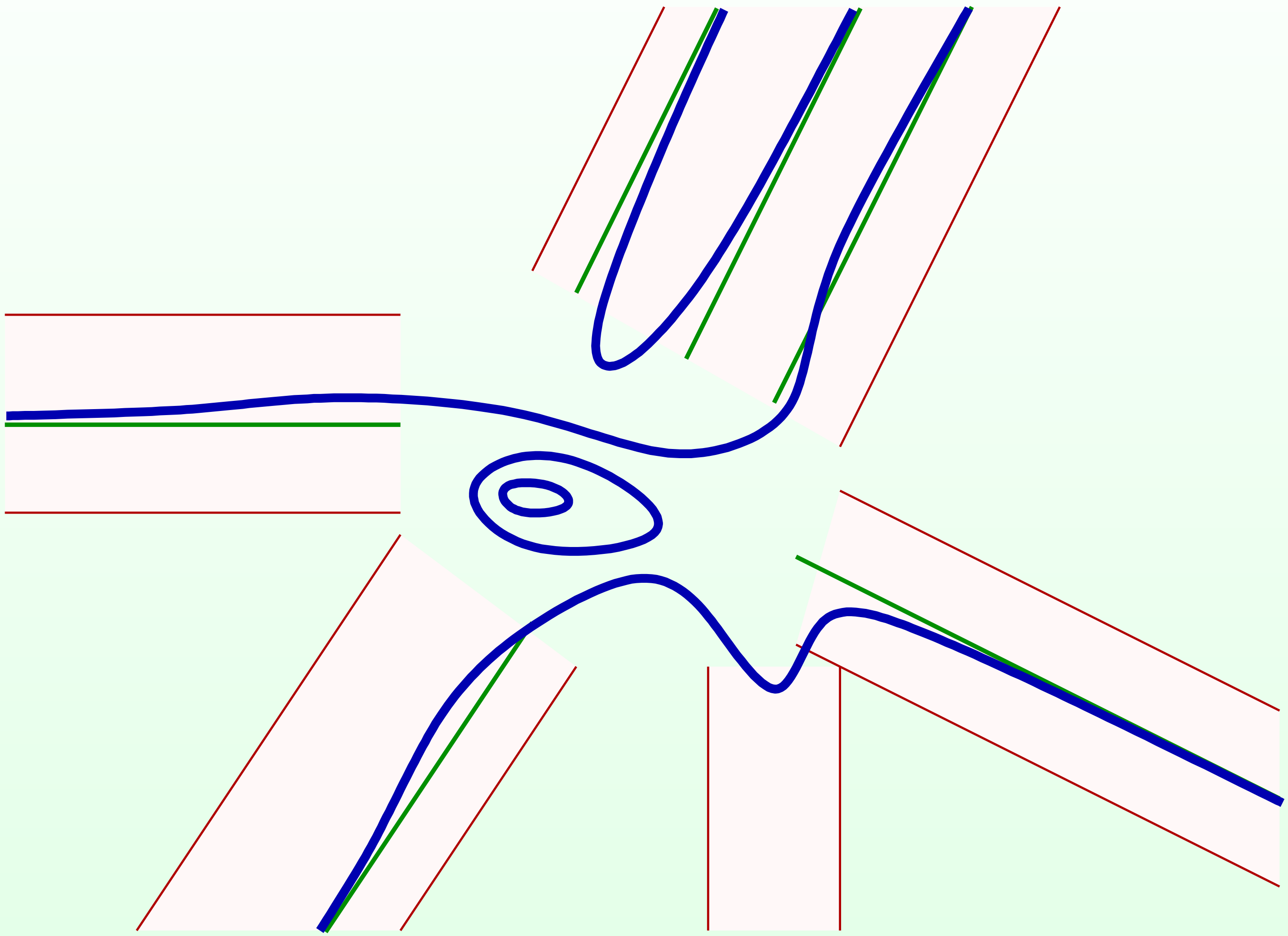
Strips in which the curve goes to the infinity.

Picture of logarithmic asymptotes



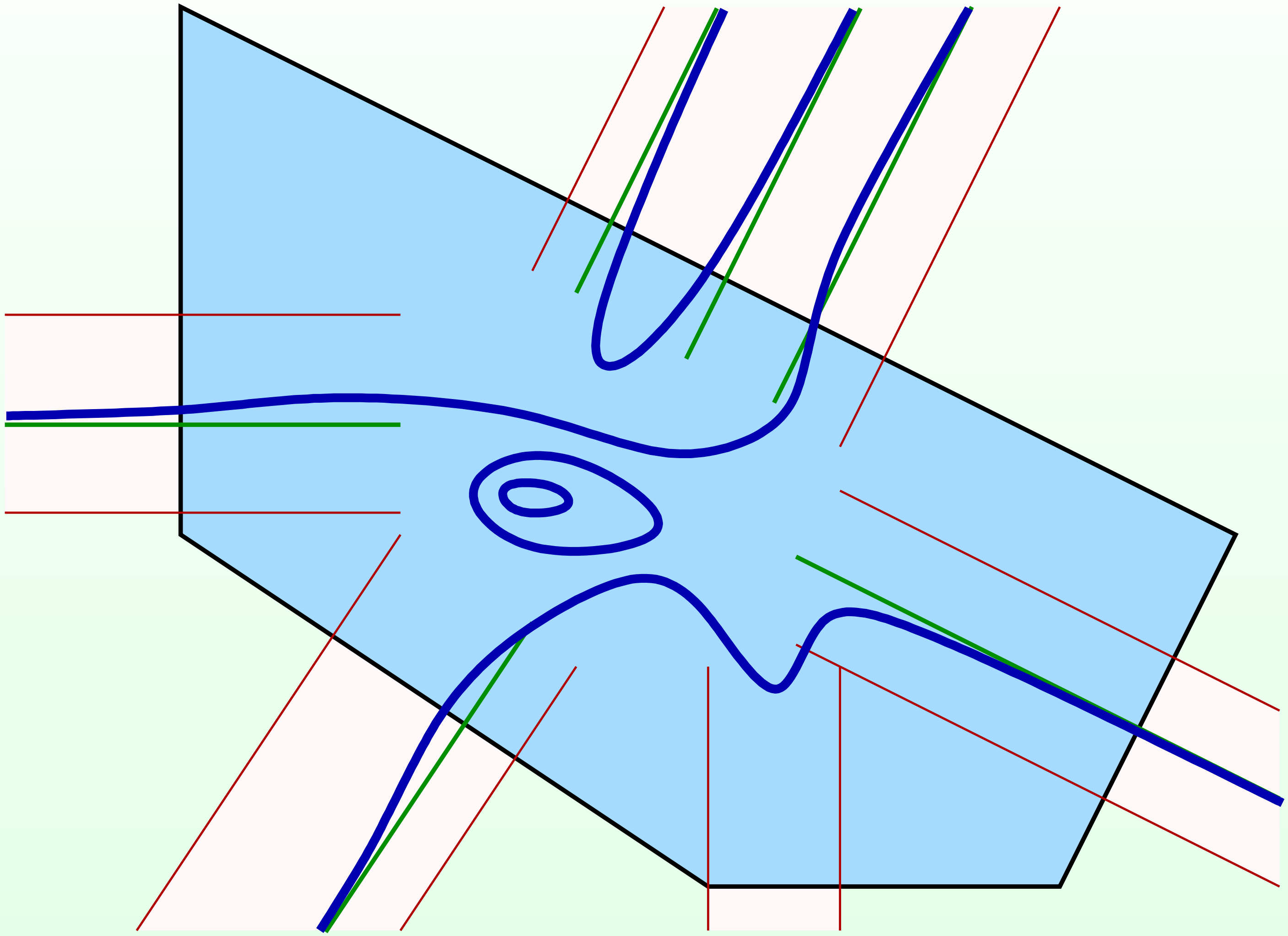
Curves defined by a^{Σ} where Σ are sides of Δ.

Picture of logarithmic asymptotes



The curve.

Picture of logarithmic asymptotes



A homothetic image of the Newton polygon intersecting the curve asymptotically stable.

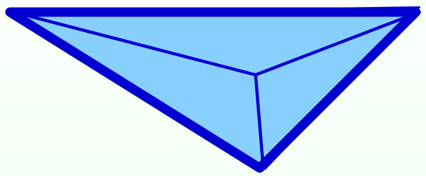
In high dimensions

everything goes similarly.

In high dimensions

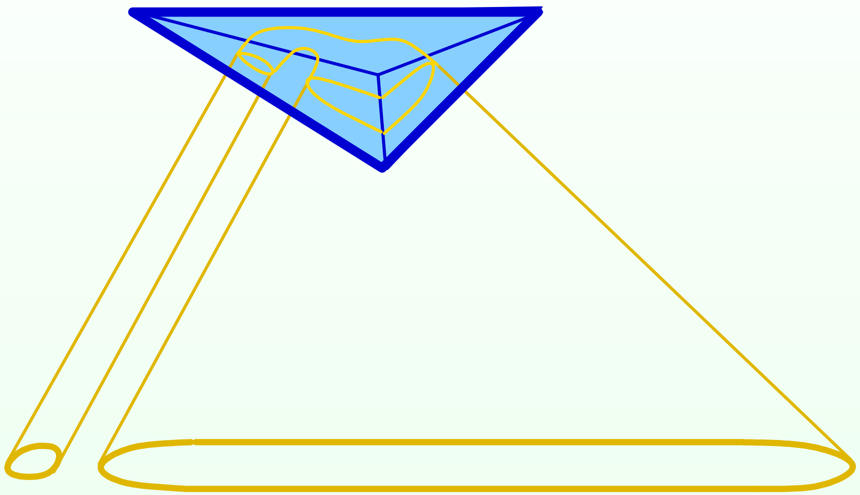
Consider a hypersurface defined by a generic polynomial

In high dimensions



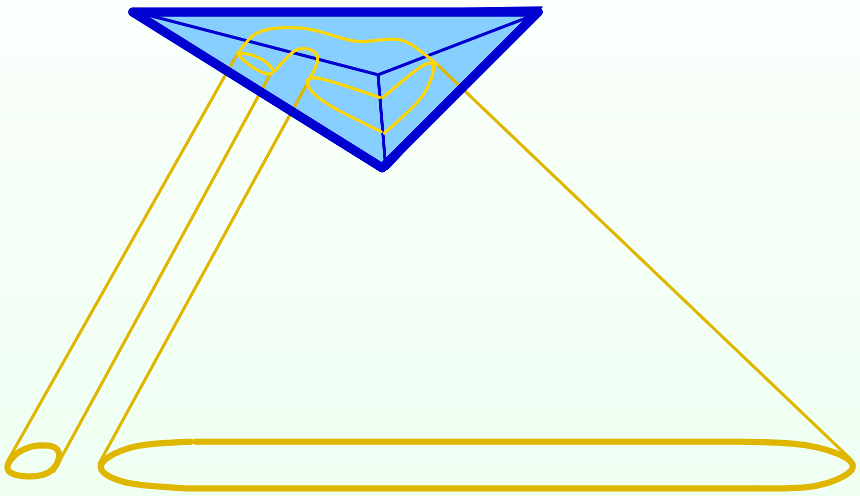
The Newton polyhedron Δ of the polynomial.

In high dimensions



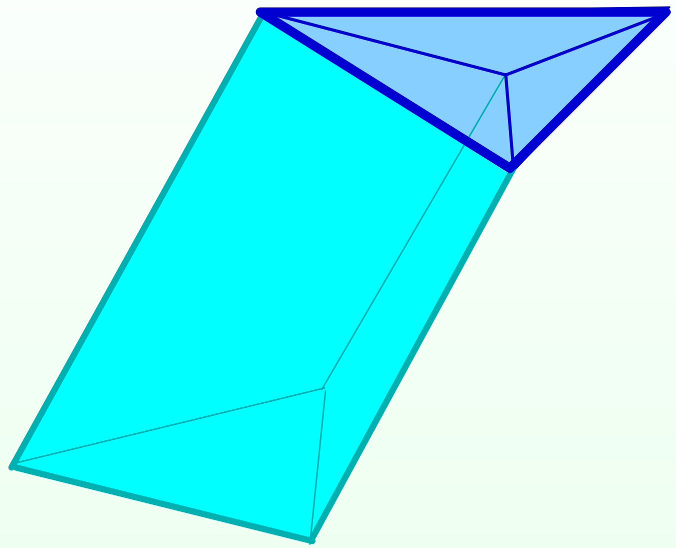
The main part of the hypersurface fits inside of sufficiently expanded Newton polyhedron.

In high dimensions



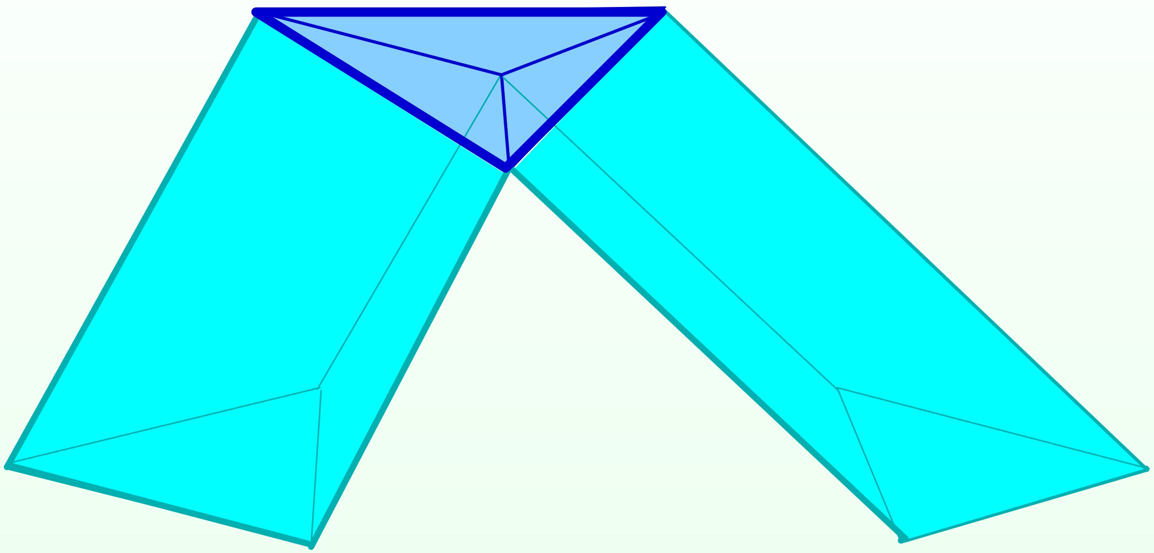
The space outside of Δ is divided into domains corresponding to the faces Δ .

In high dimensions



A prism corresponds to a principal face.

In high dimensions



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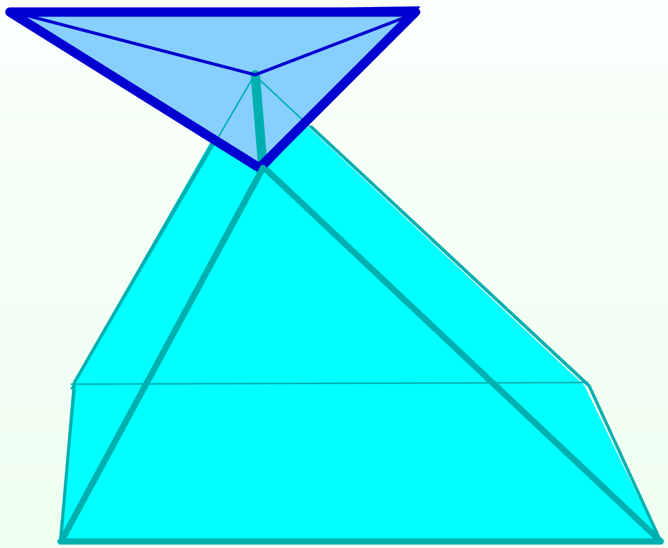
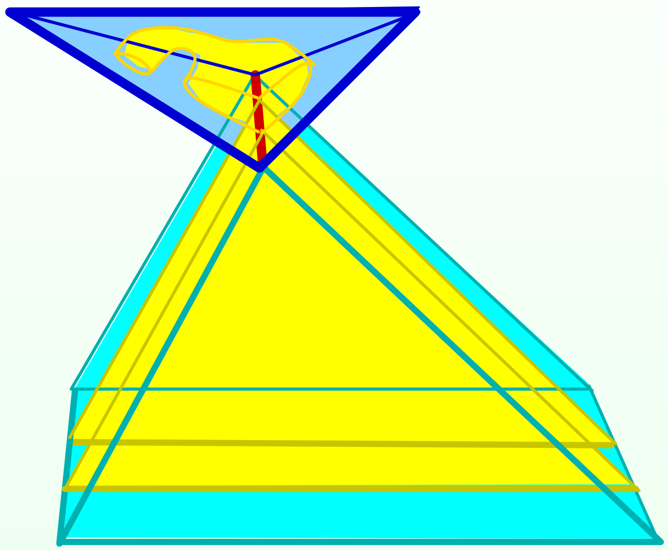


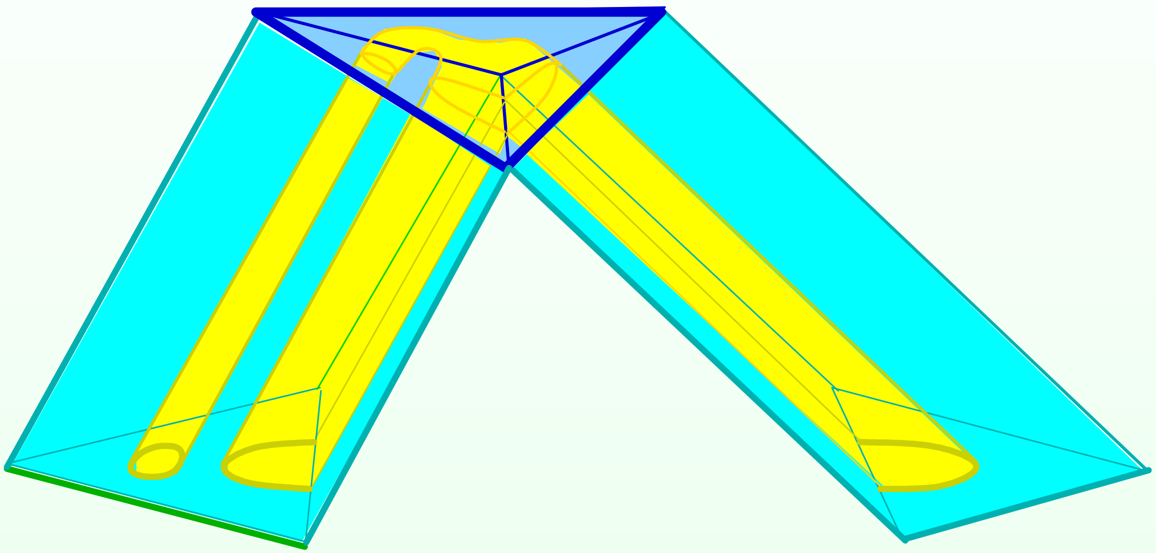
The domain corresponding to Σ has a shape of $\Sigma \times \Sigma^{\wedge}$

In high dimensions



In the domain corresponding to face Σ the hypersurface is approximated by the hypersurface defined by the part of the polynomial sitting on Σ.

In high dimensions



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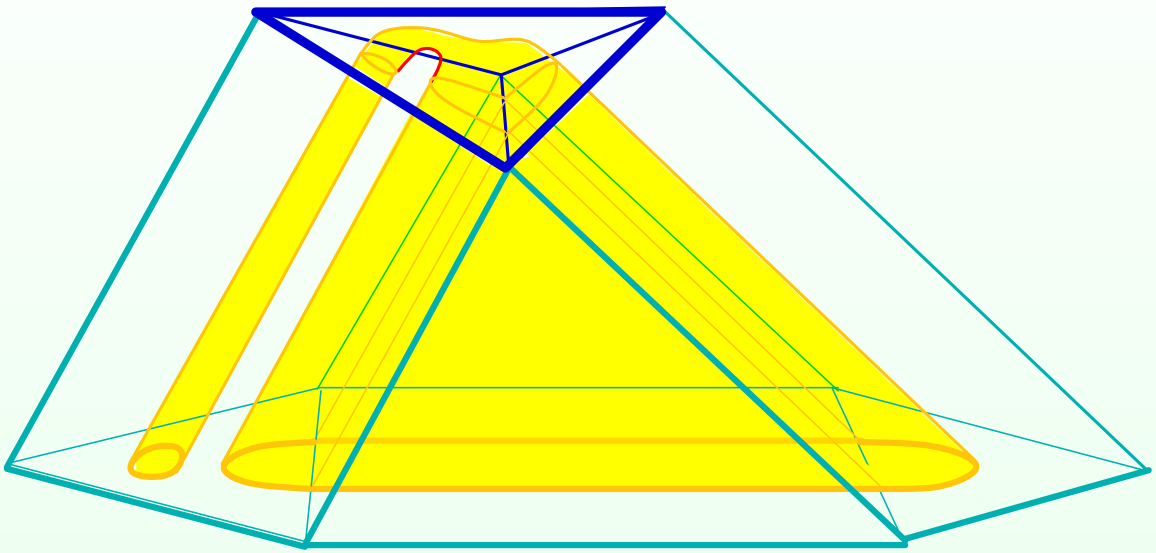

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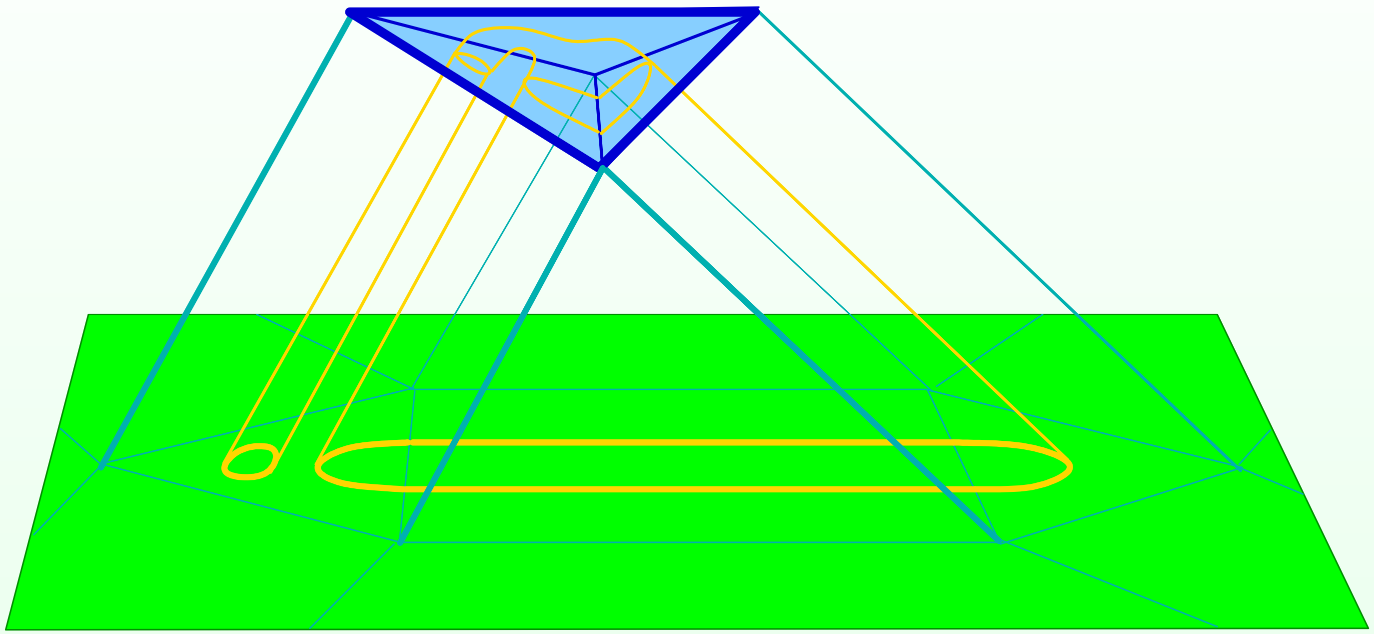


Consider a trace of the picture on a hyperplane which is bellow the Newton Polyhedron.

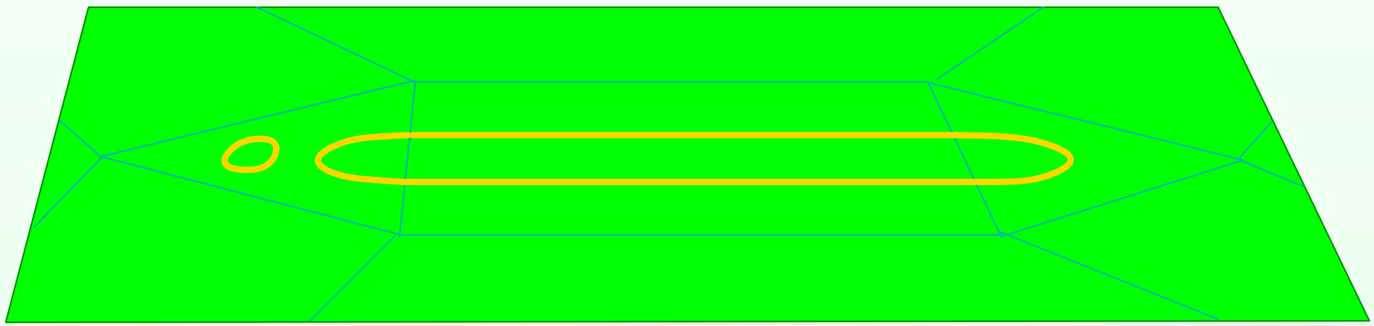
In high dimensions

The intersection of the hypersurface with the hyperplane is made of pieces corresponding to the faces of Δ looking down.

In high dimensions

The intersection of the hypersurface with the hyperplane is made of pieces corresponding to the faces of Δ looking down. This can be used to patchwork a hypersurface.

In high dimensions



The intersection of the hypersurface with the hyperplane is made of pieces corresponding to the faces of Δ looking down. This can be used to patchwork a hypersurface.
Just prepare pieces matching each other
and put them on faces of a polyhedron.

In high dimensions

The intersection of the hypersurface with the hyperplane is made of pieces corresponding to the faces of Δ looking down. This can be used to patchwork a hypersurface.
Just prepare pieces matching each other
and put them on faces of a polyhedron.
For smallest pieces it's nothing but combinatorics.

Combinatorial patchwork

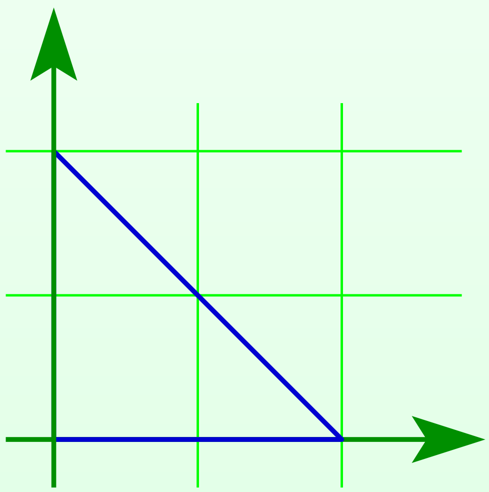
Combinatorial patchwork

Initial data for combinatorial patchworking

Combinatorial patchwork

Initial data for combinatorial patchworking

- m a positive integer (the degree of the curve),



For our example, $m = 2$.

Combinatorial patchwork

Initial data for combinatorial patchworking

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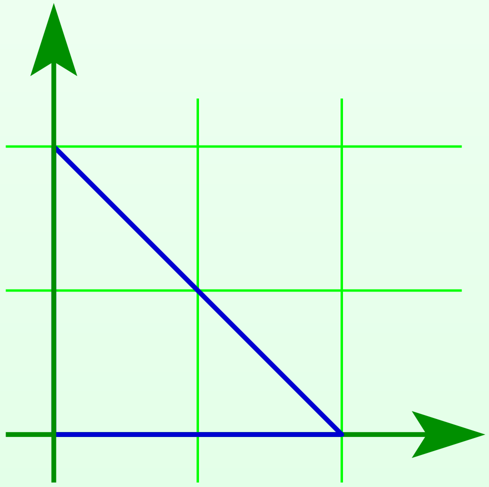
- Δ the triangle with vertices $(0,0)$, $(m,0)$, $(0,m)$,



Combinatorial patchwork

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- m a positive integer (the degree of the curve),
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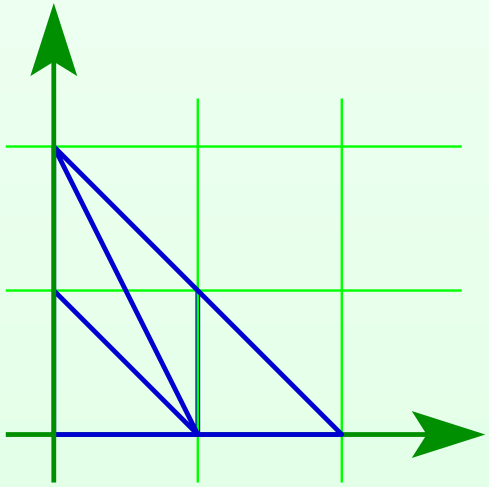

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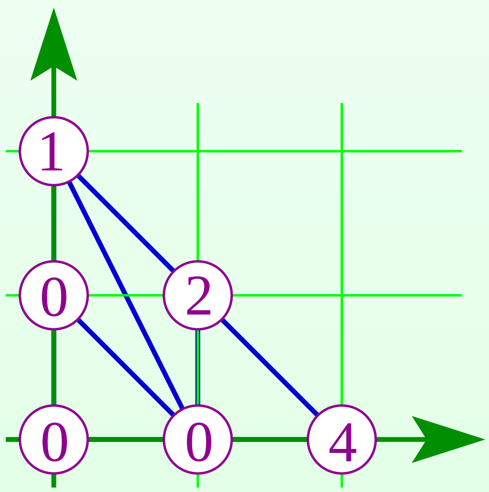
- $\nu : \Delta \longrightarrow \mathbb{R}_+$ a convex PL-function, such that triangles of τ are its domains of linearity.



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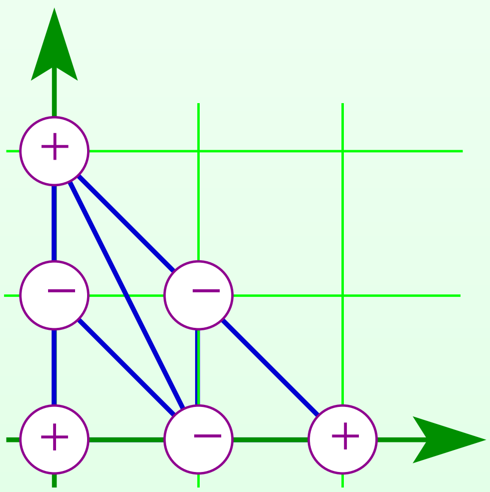
- $\sigma_{k,l}$ a sign ($+$ or $-$) at each vertex (k,l) of τ .



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Patchworking of polynomials.

Combinatorial patchwork

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Patchworking of polynomials.

$$b_t(x,y) = \sum_{\substack{(k,l) \text{ runs over} \\ \text{vertices of } \tau}} \sigma_{k,l} t^{\nu(k,l)} x^k y^l.$$

Combinatorial patchwork

Initial data for combinatorial patchworking

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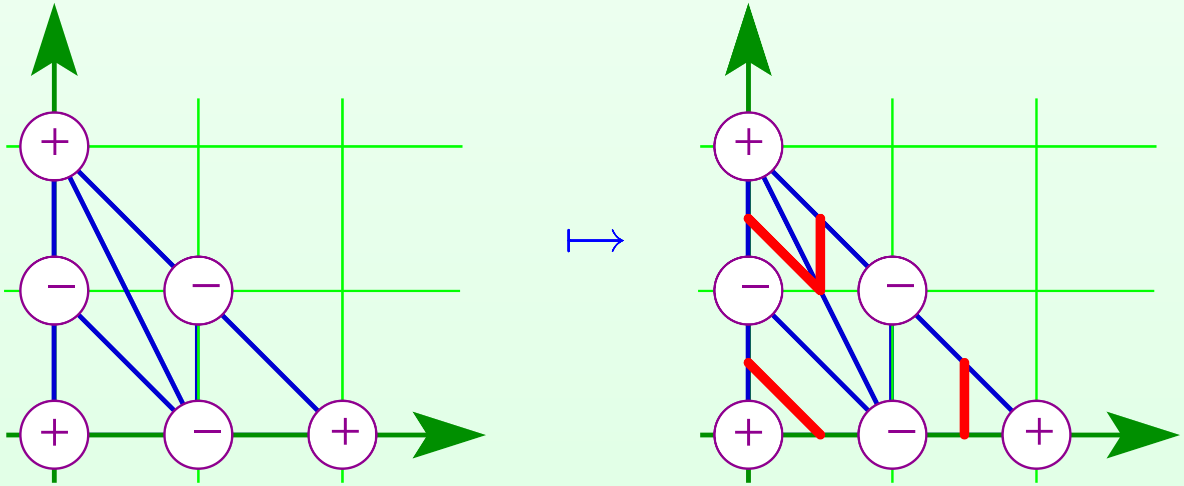
Patchworking of PL-curve.



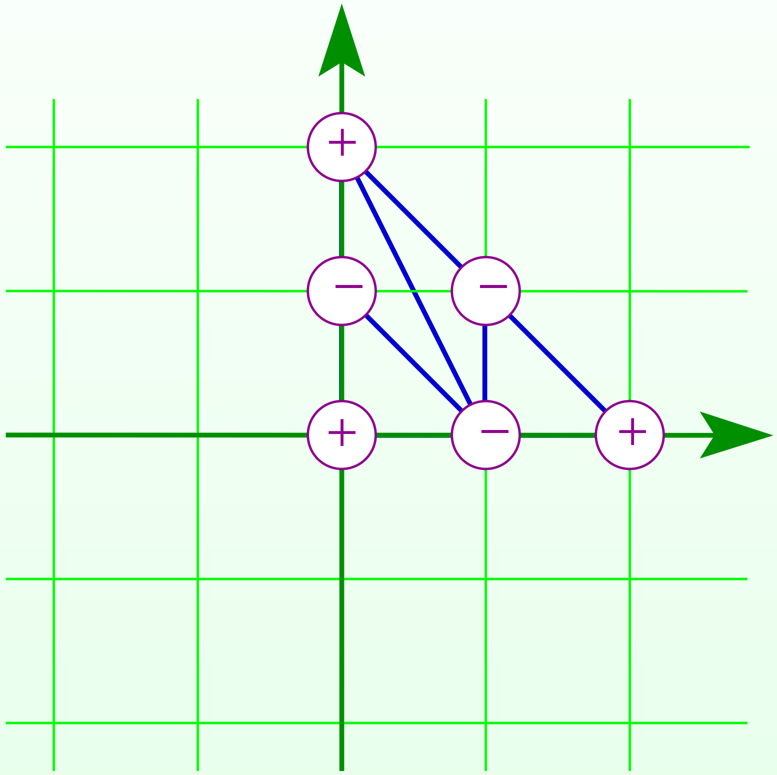
Combinatorial Patchwork Theorem

Let m, Δ, τ, $\sigma_{k,l}$ *and* ν *be initial data,*
b_t *be the patchworked polynomial*
and $L \subset \Delta$ *be the patchworked PL-curve.*

Combinatorial Patchwork Theorem

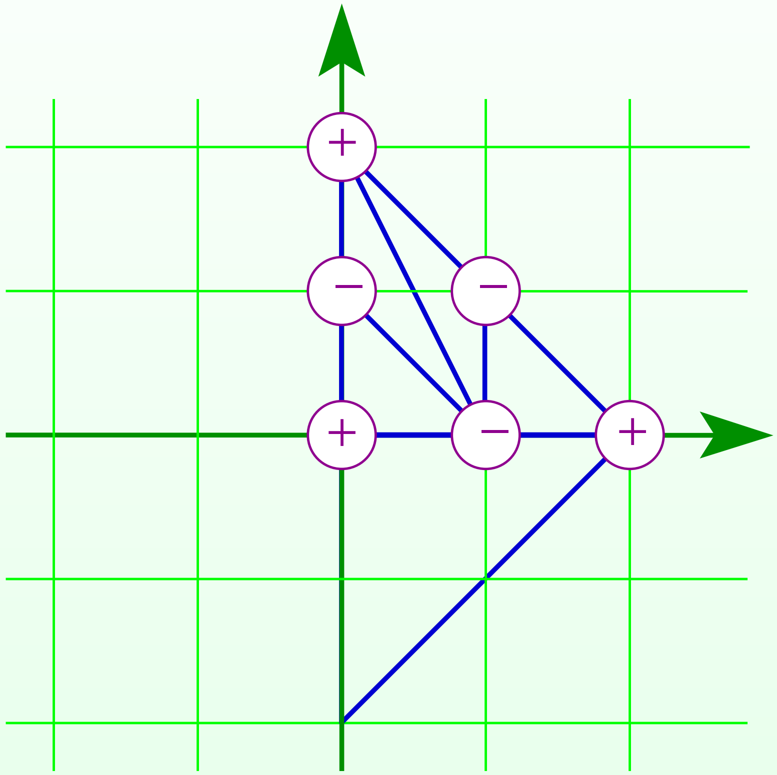
Let m, Δ, τ, $\sigma_{k,l}$ and ν be initial data,
b_t be the patchworked polynomial
and $L \subset \Delta$ be the patchworked PL-curve.
Then for sufficiently small $t > 0$ the polynomial b_t defines in the first quadrant $\mathbb{R}^2_{++} = \{(x, y) \in \mathbb{R}^2 \mid x, y > 0\}$ a curve a_t such that the pair $(\mathbb{R}^2_{++},\ a_t)$ is homeomorphic to $(\operatorname{Int}\Delta,\ L \cap \operatorname{Int}\Delta)$.

Patchwork in all quadrants



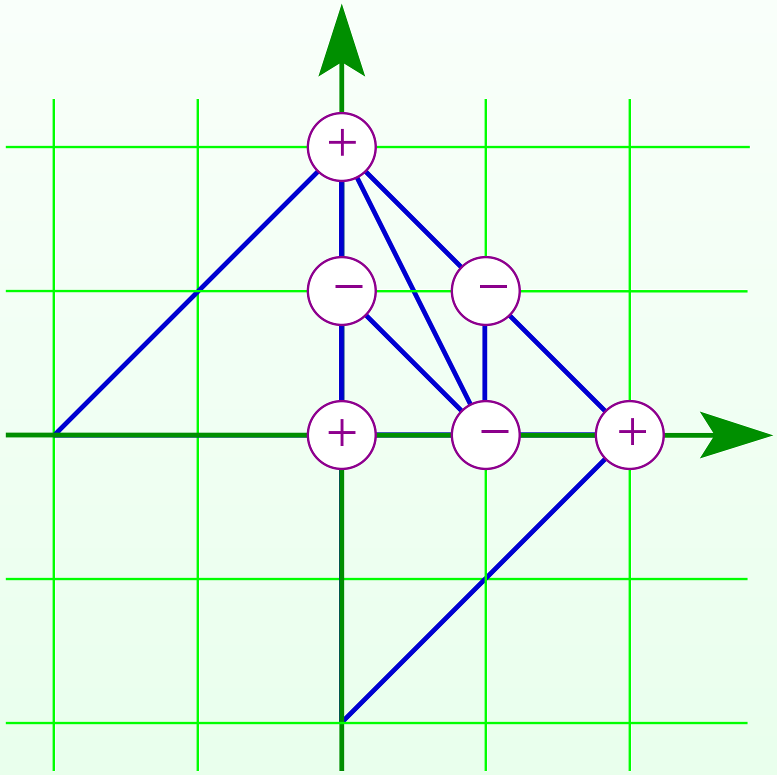
Adjoin to Δ

Patchwork in all quadrants



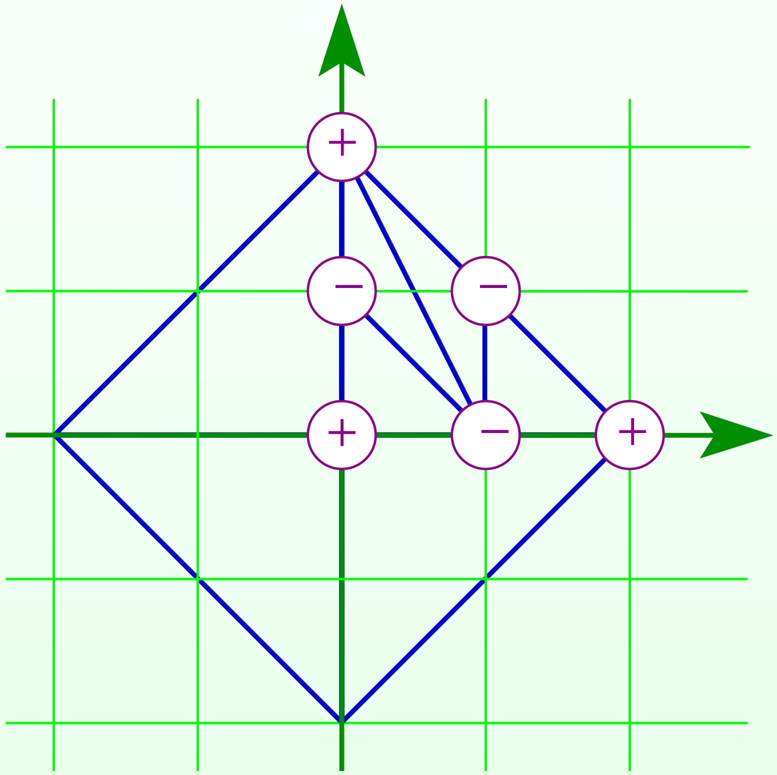
Adjoin to Δ its images $\Delta_x = s_x(\Delta)$,
where $s_x, \; s_y$ are reflections against the coordinate axes.

Patchwork in all quadrants



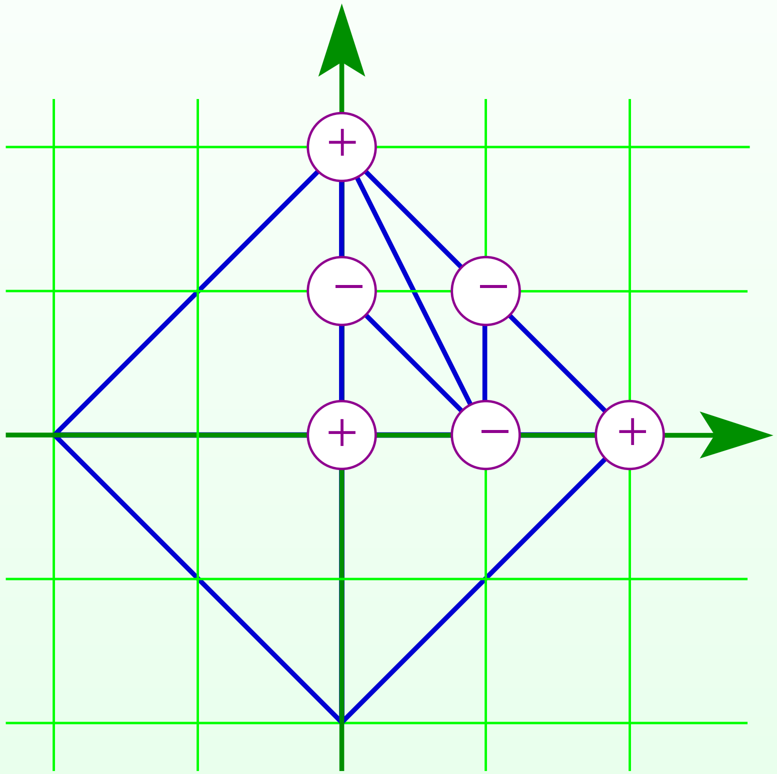
Adjoin to Δ its images $\Delta_x = s_x(\Delta)$, $\Delta_y = s_y(\Delta)$, where s_x, s_y are reflections against the coordinate axes.

Patchwork in all quadrants



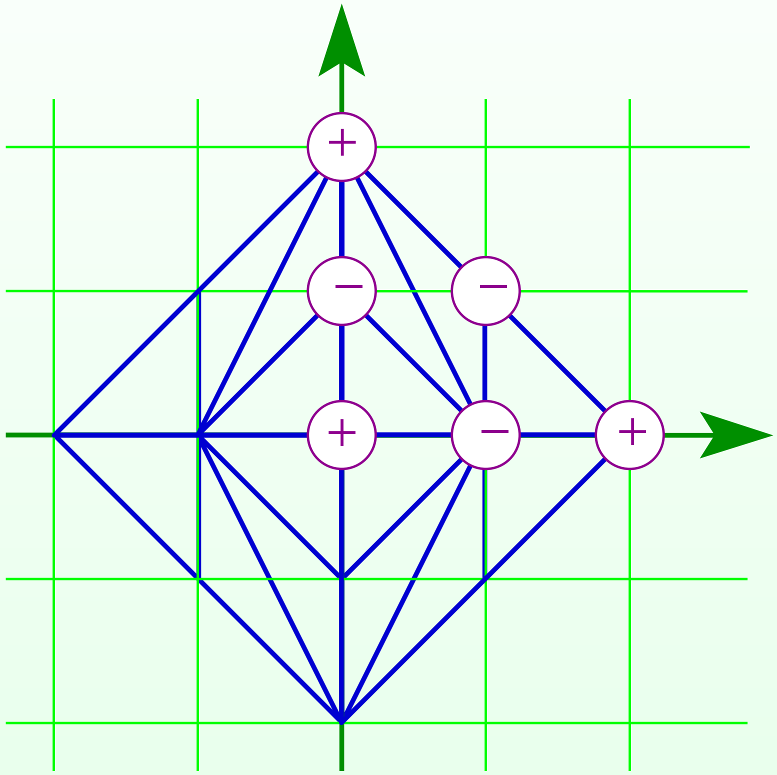
Adjoin to Δ its images $\Delta_x = s_x(\Delta)$, $\Delta_y = s_y(\Delta)$, $\Delta_{xy} = s_x \circ s_y(\Delta)$,
where $s_x,\ s_y$ are reflections against the coordinate axes.

Patchwork in all quadrants



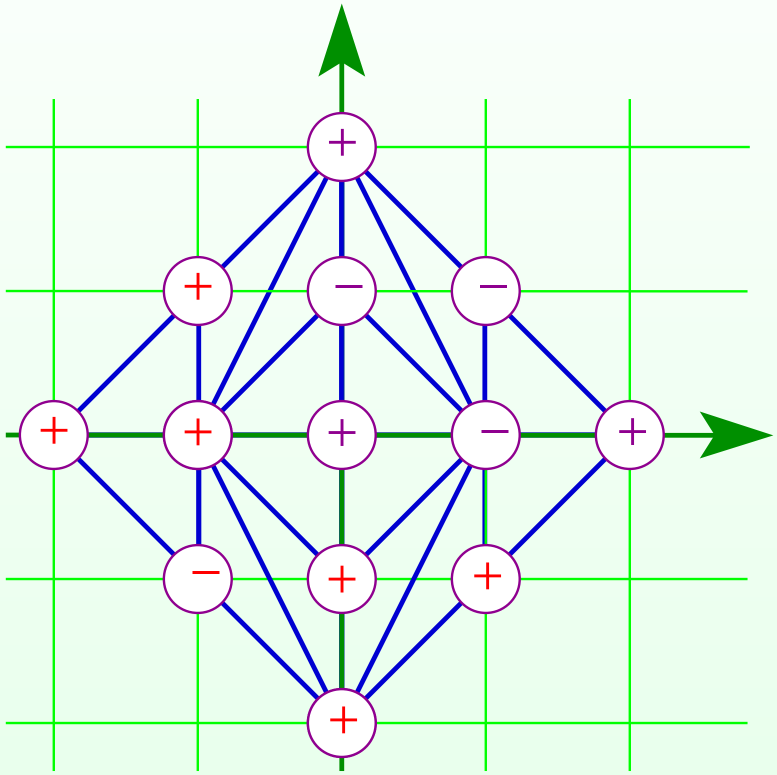
Put $A\Delta = \Delta \cup \Delta_x \cup \Delta_y \cup \Delta_{xy}$.

Patchwork in all quadrants



Extend τ to a symmetric triangulation $A\tau$ of $A\Delta$,

Patchwork in all quadrants



Extend $\sigma_{i,j}$ to a distribution of signs at the vertices of $A\tau$ by the rule: $\sigma_{i,j}\sigma_{\epsilon i,\delta j}\epsilon^i\delta^j = 1$, where $\epsilon, \delta = \pm 1$.

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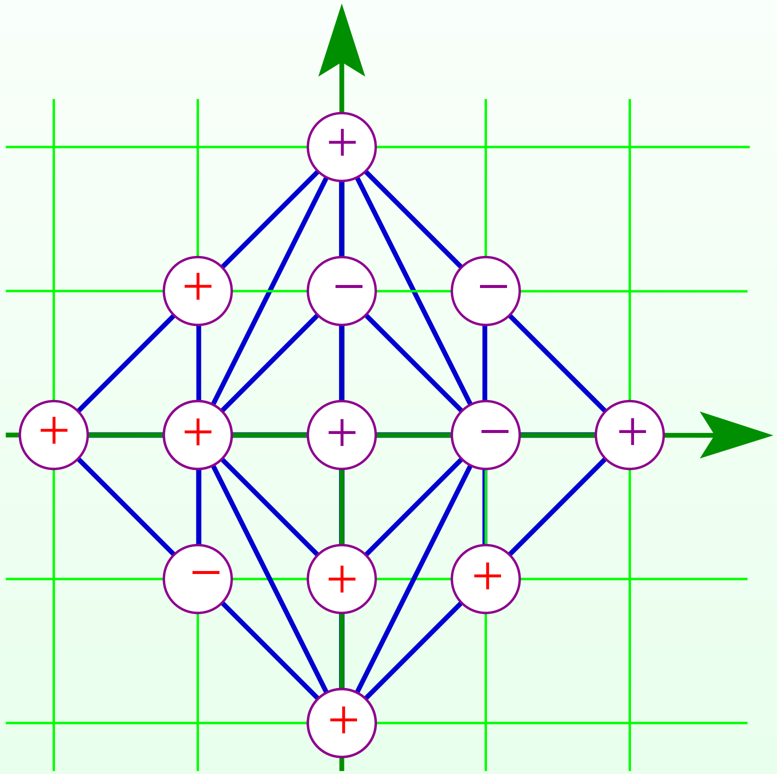

(In other words, passing from a vertex to its mirror image with respect to an axis we preserve its sign if the distance from the vertex to the axis is even, and change the sign otherwise.)

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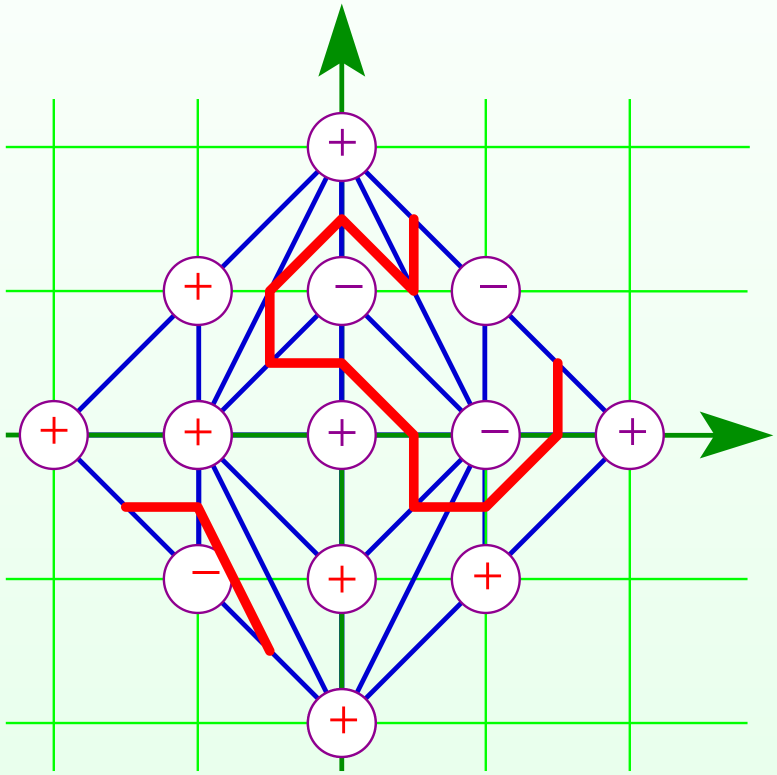
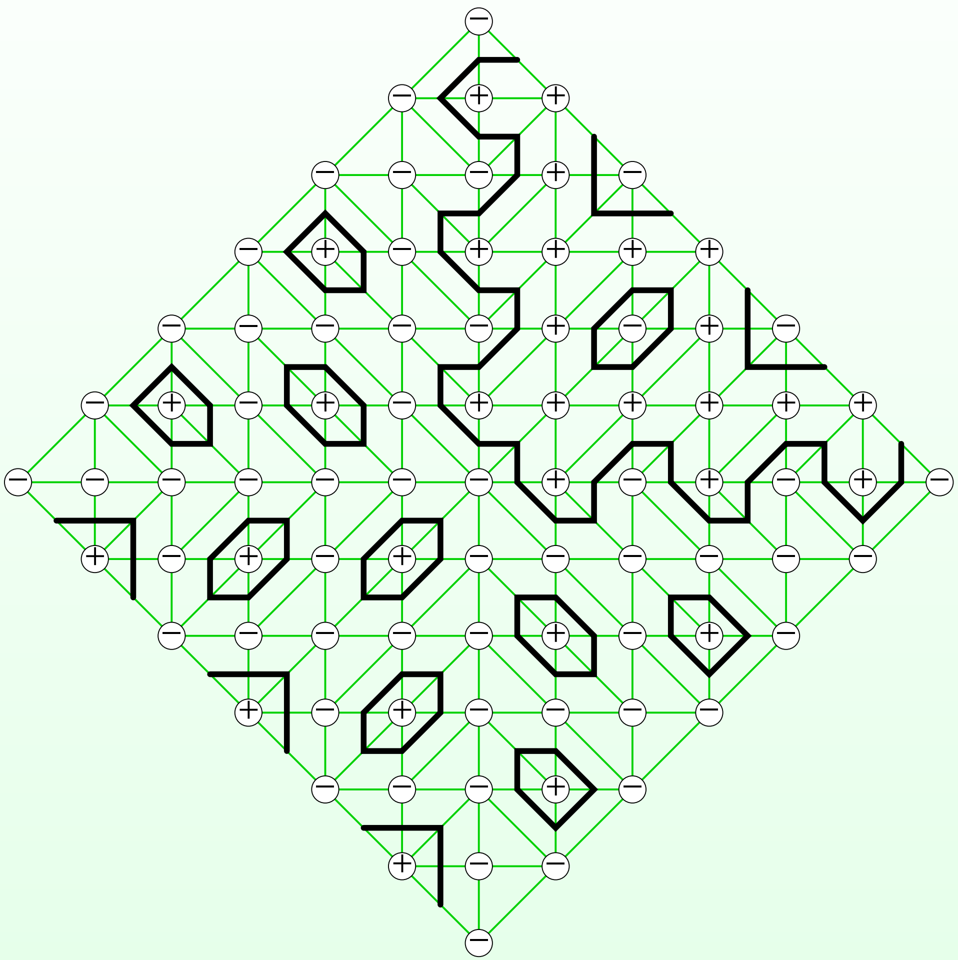


Draw the midlines.

Addendum to the Patchwork Theorem.

Under the assumptions of Patchwork Theorem, for all sufficiently small $t > 0$ *there exist a homeomorphism* $A\Delta \to \mathbb{R}^2$ *mapping* AL *onto the the affine curve defined by* b_t *and a homeomorphism* $P\Delta \to \mathbb{R}P^2$ *mapping* PL *onto the projective closure of this affine curve.*

Patchworking of the Harnack curve of degree 6



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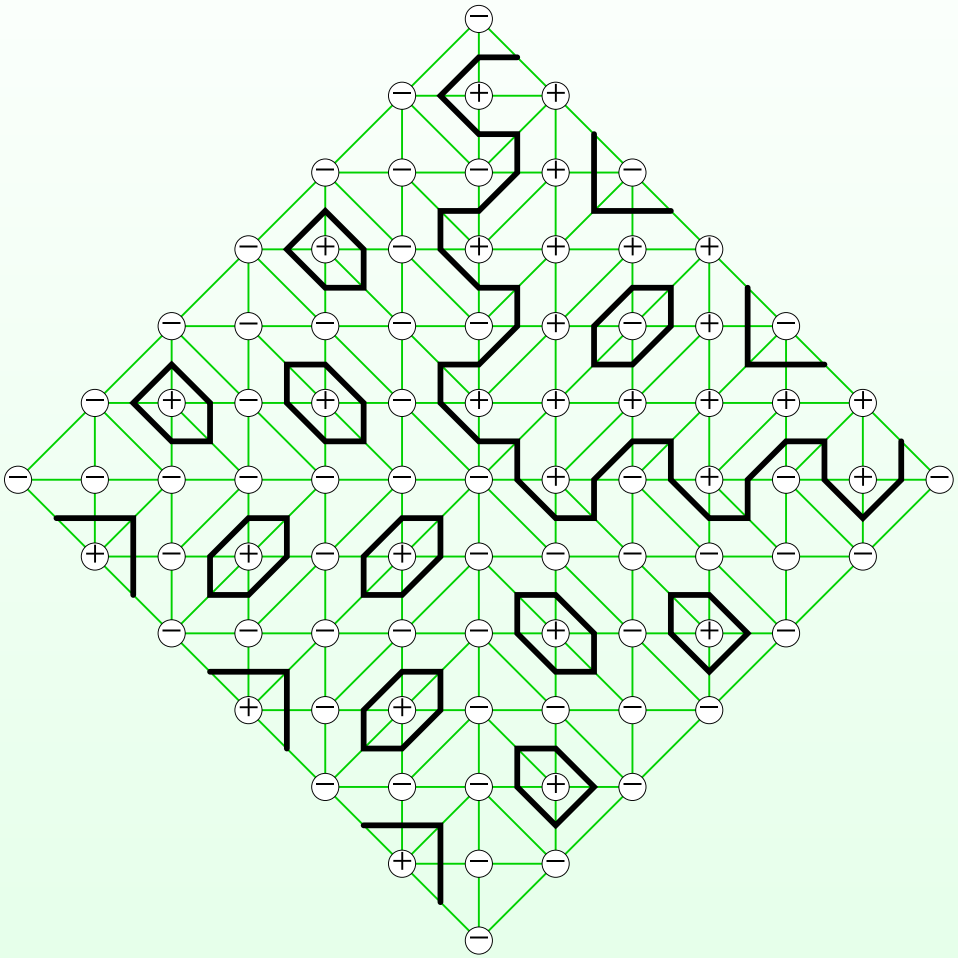
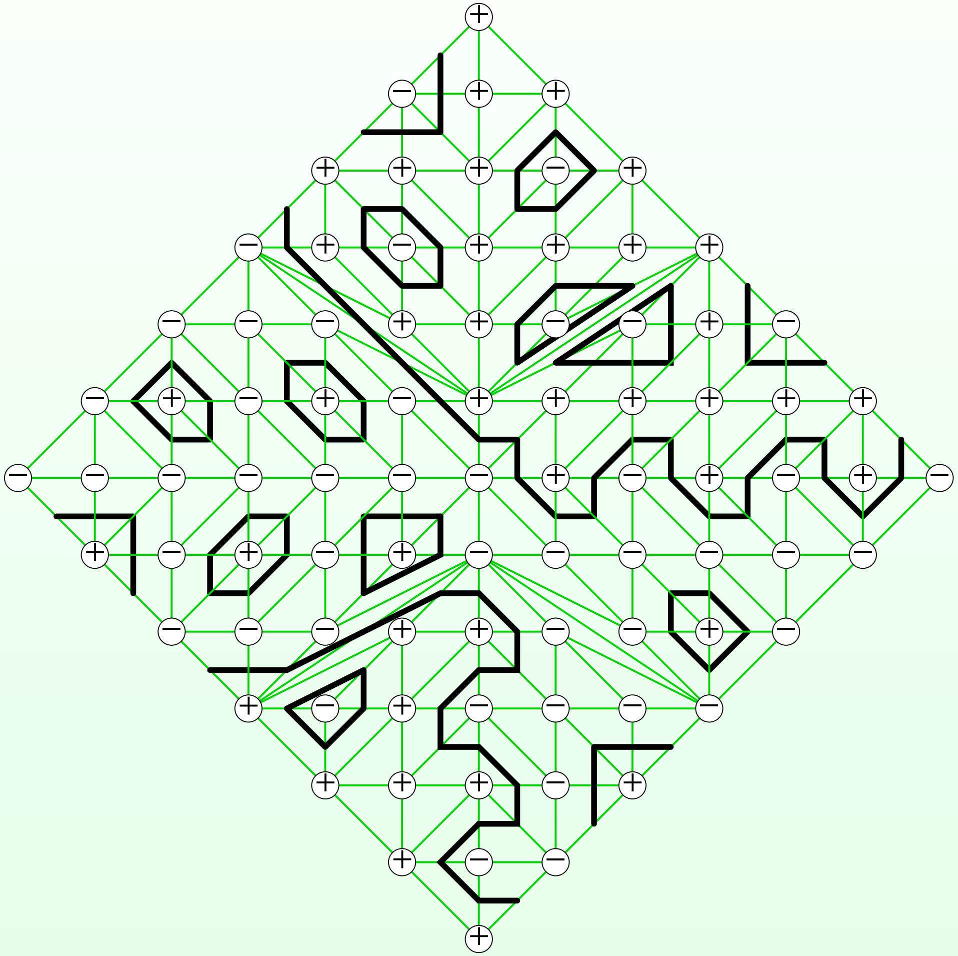


Nine empty ovals and two nested ovals.

Gudkov's curve



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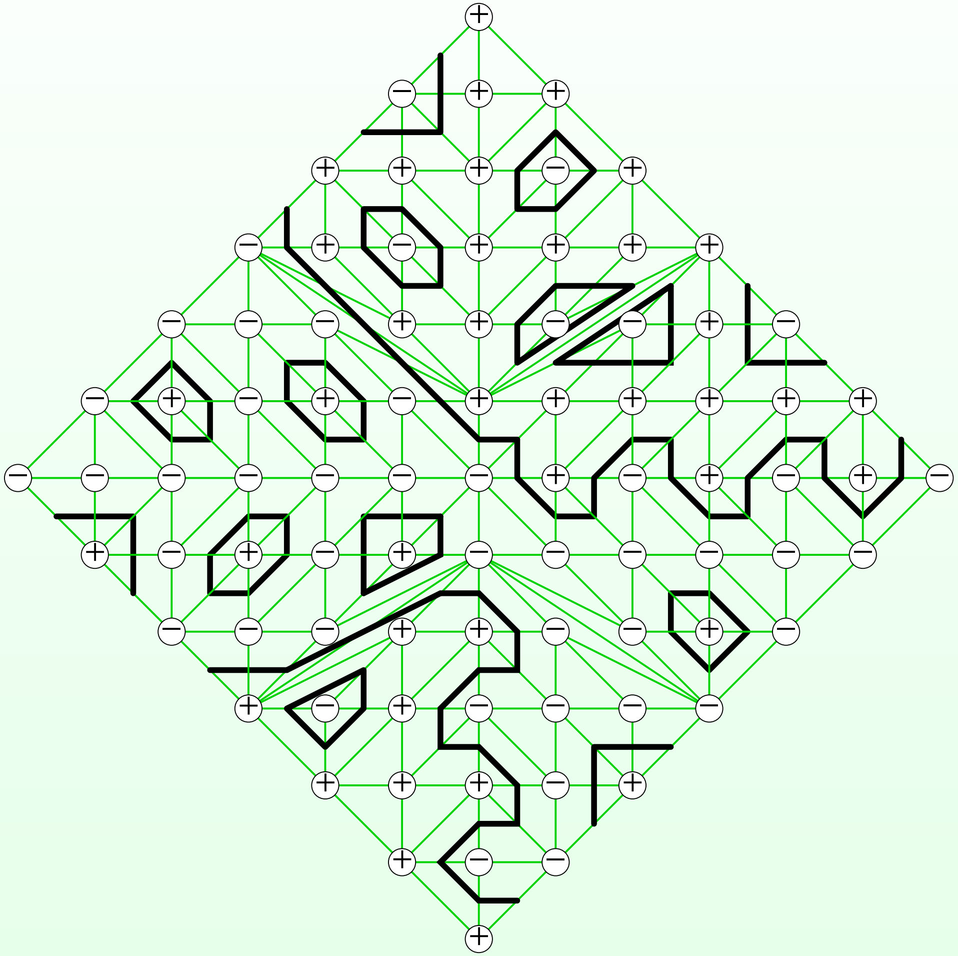
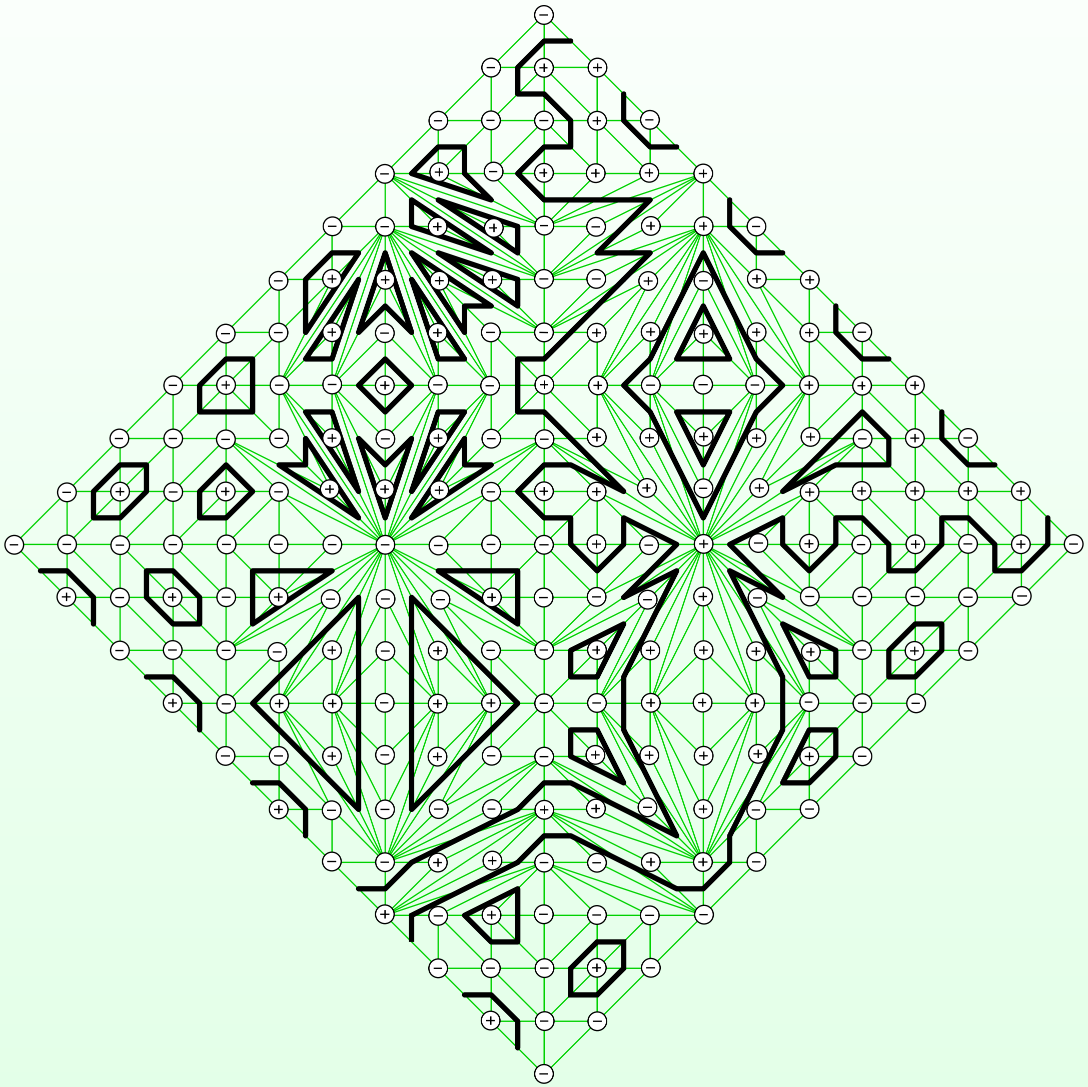


Patchworking of the Gudkov curve of degree 6. Five empty ovals and an oval enclosing five other empty ovals.

Curve of degree 10 with 32 odd ovals



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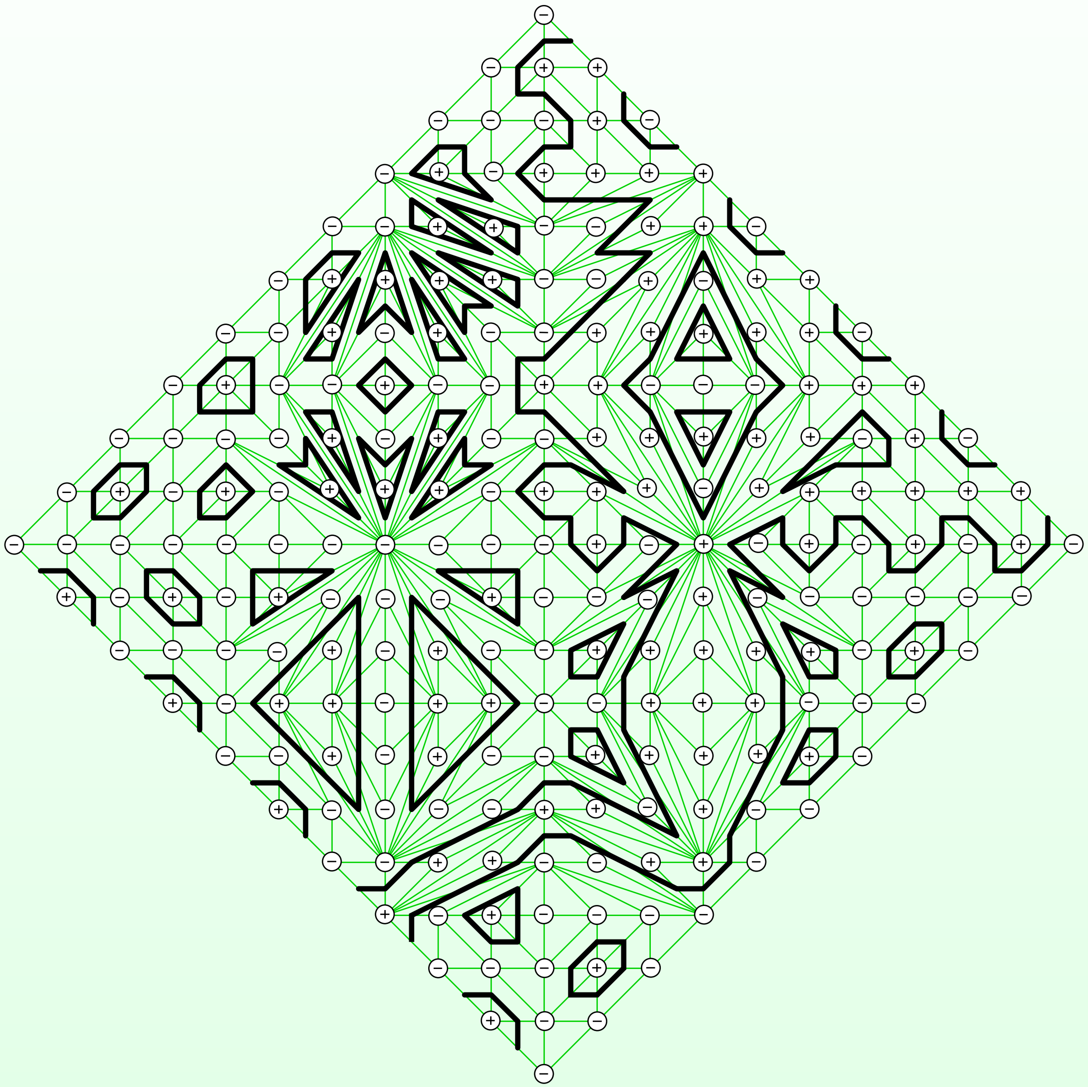

Ilia Itenberg's patchworking of a counter-example to the Ragsdale Conjecture. A curve of degree 10 with 32 odd ovals.

Tropical

Arnold's advice

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I could not find any relation, but was not disappointed.

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Correspondence Principle (G. L. Litvinov and V. P. Maslov)
"There exists a (heuristic) correspondence, in the spirit of the correspondence principle in Quantum Mechanics, between important, useful and interesting constructions and results over the field of real (or complex) numbers (or the semiring of all nonnegative numbers) and similar constructions and results over idempotent semirings."

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The deformation consists of the graphs of the same polynomial $\sum_k \ln(a_k)x^k$*, but on* S_h^2 *with varying* $h \in [0,1]$.

Real algebraic geometry as quantized PL-geometry

$$\text{Real polynomials} \xleftarrow{\text{quantization}} \text{Convex PL-functions with integral slopes}$$

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$$\begin{array}{ccc} \text{Real polynomials} & \xleftarrow{\text{quantization}} & \text{Convex PL-functions with integral slopes} \\ \Big\downarrow \text{generate} & & \text{generate} \Big\downarrow \\ & & \end{array}$$

Real algebraic geometry as quantized PL-geometry

$$\begin{tikzcd}[column sep=huge, row sep=large]
\begin{array}{c}\text{Real}\\ \text{polynomials}\end{array} \arrow[d, "\text{generate}"] & \begin{array}{c}\text{Convex}\\ \text{PL-functions}\\ \text{with integral slopes}\end{array} \arrow[l, "\text{quantization}"'] \arrow[d, "\text{generate}"'] \\
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I presented this in my talk at European Congress of Mathematicians in 2000.

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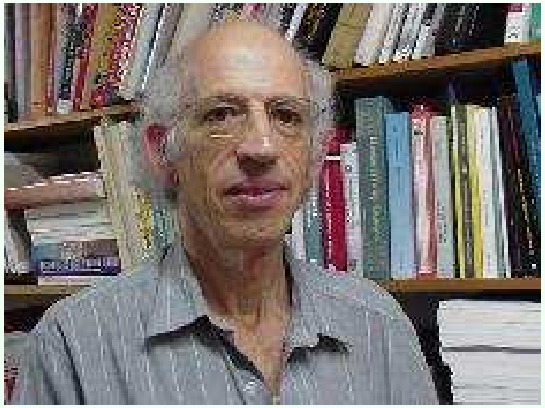

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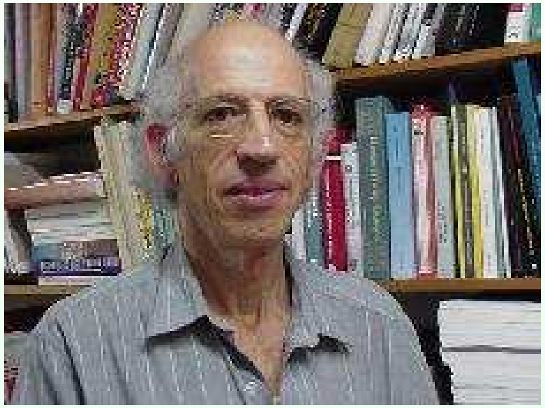

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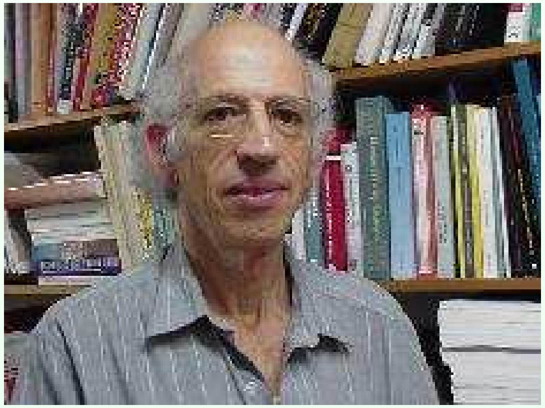

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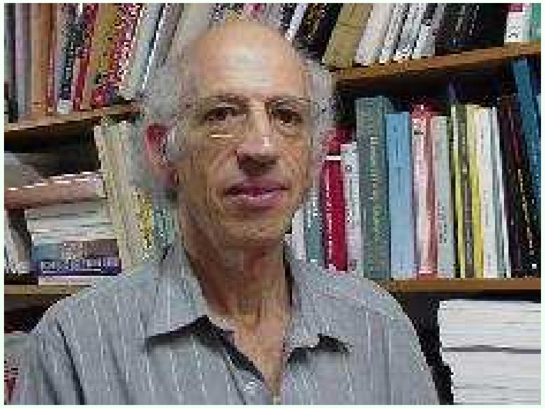

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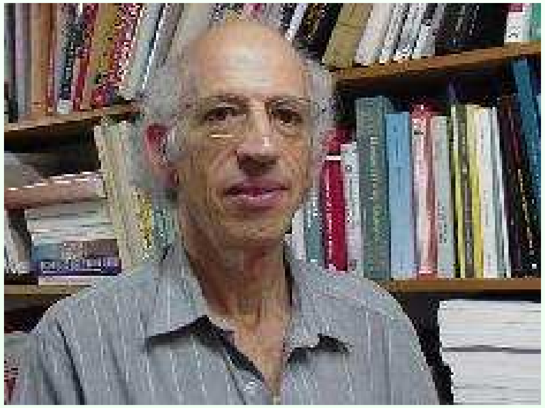

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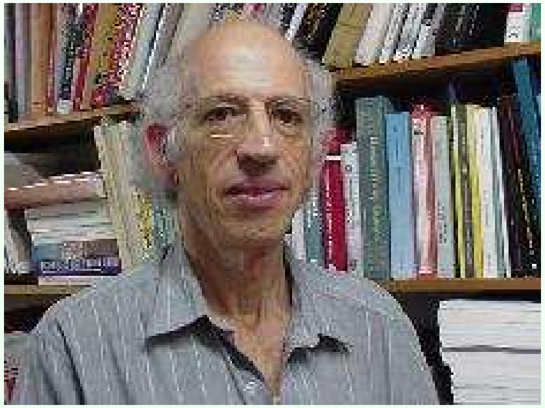

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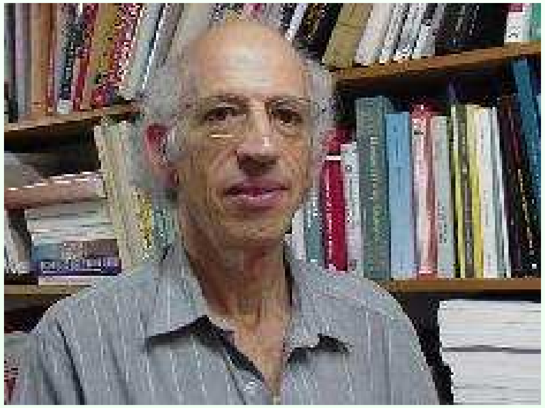

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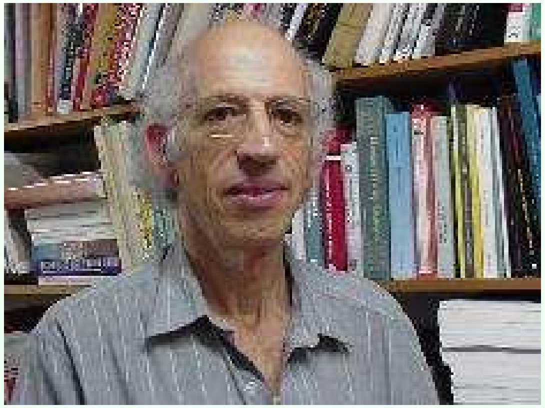

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Applications (besides combinatorial patchworking) in enumerative geometry, both real and complex.