Hypergeometries. I

Oleg Viro

August 20, 2013

Hyperalgebra

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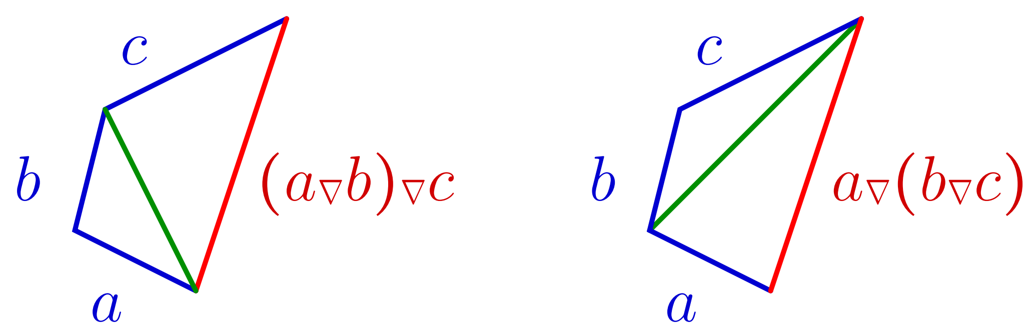

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$\mathbb{R}_{\geq 0}$ with addition $(a, b) \mapsto a \triangledown b$ and usual multiplication is a **hyperfield**.

Hyperfields

A set X with a multivalued operation

$$X \times X \to 2^X \setminus \{\varnothing\} : (a, b) \mapsto a \top b$$

and a multiplication $X \times X \to X : (a, b) \mapsto a \cdot b$ is called a **hyperfield**, if

- $(a, b) \mapsto a \top b$ is commutative, associative;
- $\exists 0 \in X$ such that $0 \top a = a$ for any $a \in X$;
- for $\forall a \in X$ there exists a unique $-a \in X$ such that $0 \in a \top (-a)$;
- $-(a \top b) = (-a) \top (-b)$
- $0 \cdot a = a \cdot 0 = 0$ for any $a \in X$;
- distributivity: $a(b \top c) = ab \top ac$ for any $a, b, c \in X$;
- $X \setminus 0$ is a commutative group under the multiplication.

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The **sign hyperfield**: $\mathbf{S} = \{0, 1, -1\}$ with $1 \smile 1 = 1$, $-1 \smile -1 = -1$,

$1 \smile -1 = \{1, 0, -1\}$.

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For any hyperfield X, $(n \times n)$-matrices with elements from X and with obvious operations form a hyperring.

Hyperring homomorphisms

A map $f : X \to Y$ is called a (hyperring) homomorphism if

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4. $$\operatorname{sign} : \mathbb{R} \to \mathbf{S} : x \to \begin{cases} +1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0. \end{cases}$$ is a hyperring homomorphism.

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If X is a hyperfield, $h : X \to Y$ a hyperring homomorphism,
then $\operatorname{Ker} h$ is an ideal in X and hence $\operatorname{Ker} h = 0$.

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For hyper**rings** the notion of ideal

should be borrowed from Berkovich's $\mathbb{F}_1$ category.

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Ultrametric = isosceles with legs not shorter than the base.

The amoeba hyperfield

Another view on the triangular hyperfield $\triangledown$:

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Another view on the triangular hyperfield ∇ :

By the bijection

$$\mathbb{R} \cup \{-\infty\} \to \mathbb{R}_{\geq 0} : \begin{cases} x \mapsto \log x, & \text{for } x \neq -\infty \\ -\infty \mapsto 0 \end{cases}$$

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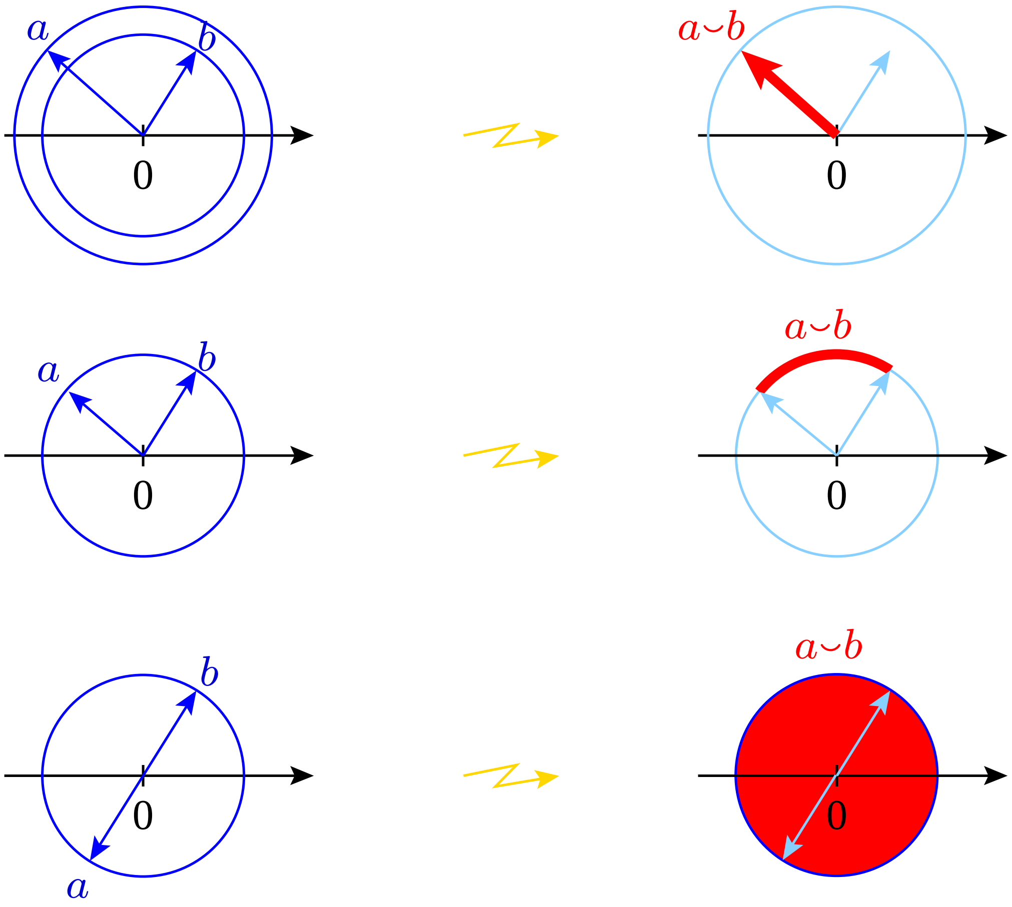
The hyperfield gotten as the result is called the **amoeba hyperfield** and denoted by $\mathcal{A}$.

Tropical addition of complex numbers



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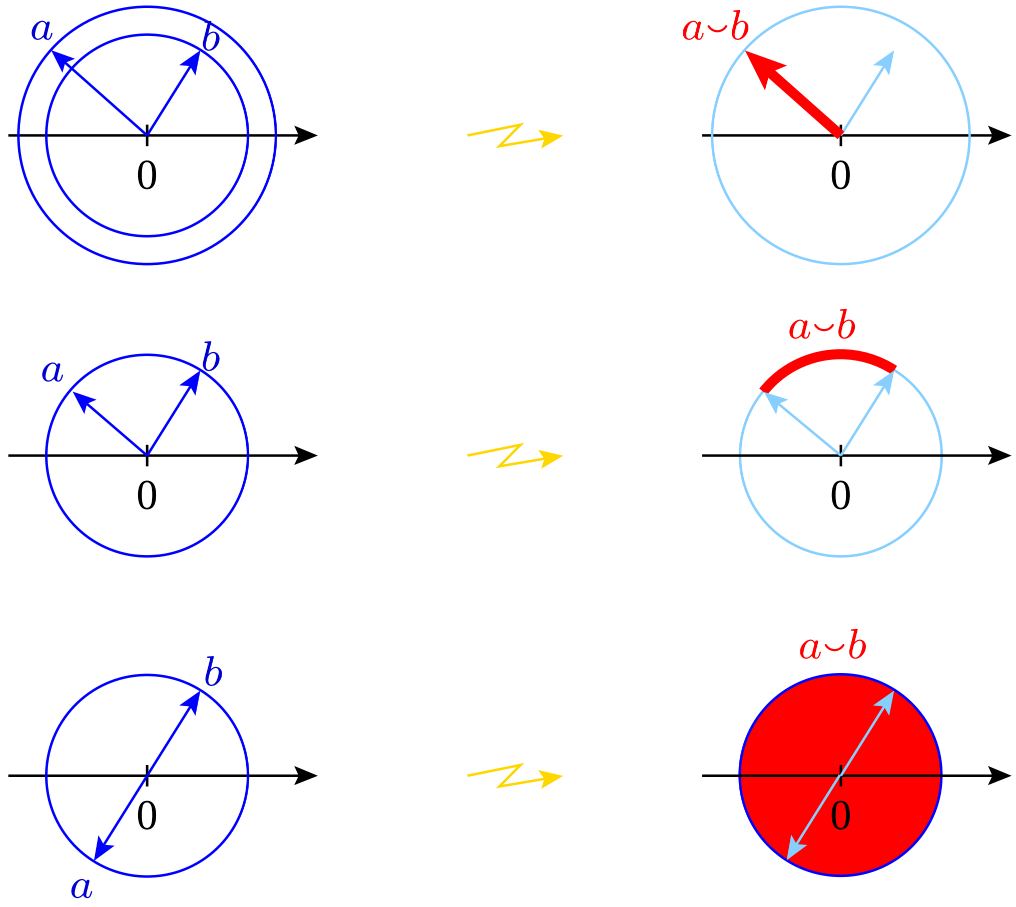

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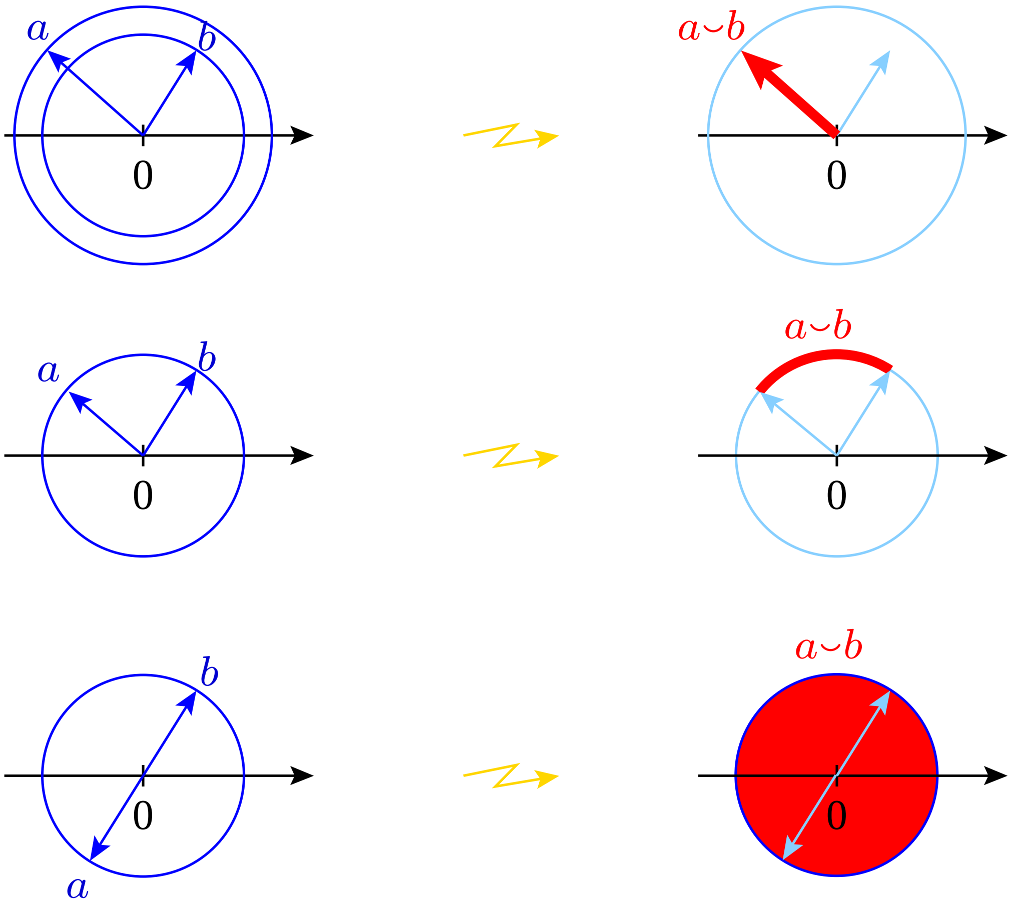

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The **complex tropical hyperfield** $\mathcal{T}\mathbb{C}$.

Properties of tropical addition

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How do several complex numbers with the same absolute values give zero?

Properties of tropical addition

$0 \in a \smile b \smile c \smile \ldots \smile x$ iff $0 \in \mathrm{Conv}(a, b, c, \ldots, x)$.

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What if the summands have different absolute values?

Then only those with the greatest one matter!

Properties of tropical addition

$0 \in a \smile b \smile c \smile \ldots \smile x$ iff $0 \in \mathrm{Conv}(a, b, c, \ldots, x)$.

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If p is a complex tropical polynomial and $X \subset \mathbb{C}$ is a closed set, then $p^{-1}(X) = \{a \mid X \subset p(a)\}$ is closed.

Tropical addition of real numbers

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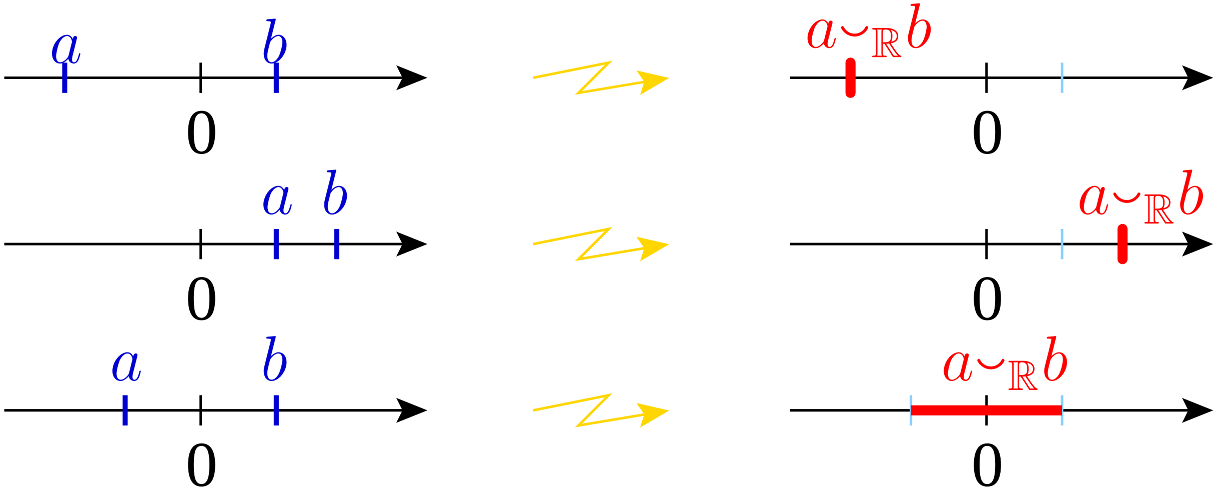

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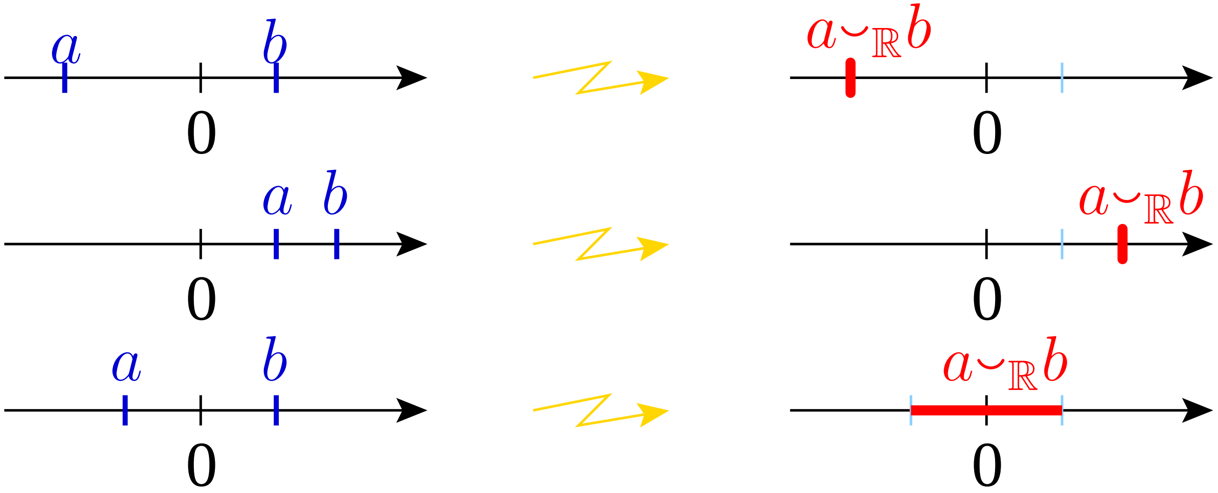

For $a, b \in \mathbb{R}$

$$a \smile_{\mathbb{R}} b = \begin{cases} \{a\}, & \text{if} \quad |a| > |b|, \\ \{b\}, & \text{if} \quad |a| < |b|, \\ \{a\}, & \text{if} \quad a = b, \\ [-|a|, |a|], & \text{if} \quad a = -b. \end{cases}$$

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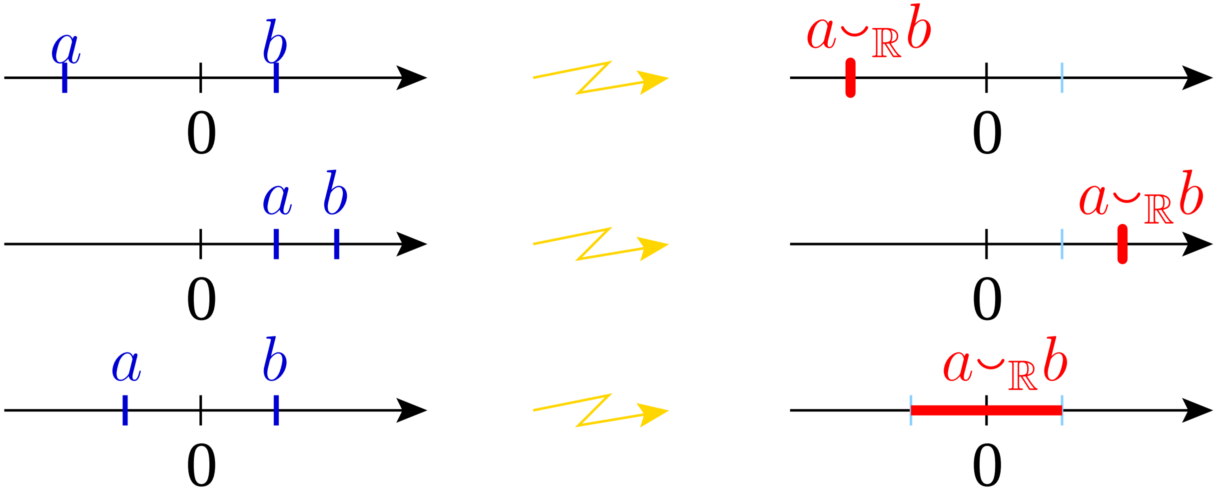

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Theorem. $\mathcal{T}\mathbb{R} = (\mathbb{R}, \smile_{\mathbb{R}}, \times)$ is a hyperfield.

Other subhyperfields of $\mathcal{T}\mathbb{C}$

The sign hyperfield $\mathbf{S} = \{0, 1, -1\}$ is a subhyperfield of $\mathcal{T}\mathbb{R} \subset \mathcal{T}\mathbb{C}$.

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According to Connes and Consani, $a \in a \top a$ means **characteristic one.**

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Theorem. Any $X \subset \mathbb{C}$ containing 0 and 1,
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The inclusion $(\mathbb{R}_{\geq 0}, \max, \times) \hookrightarrow \mathcal{T}\mathbb{R}$ is a homomorphism.

Hyperring homomorphisms

Hyperring is a hyperfield with no division required.
A map $f: X \to Y$ is called a (hyperring) homomorphism if $f(a \top b) \subset f(a) \top f(b)$ and $f(ab) = f(a)f(b)$ for any $a, b \in X$.

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A multiplicative non-archimedean norm $K \to \mathbb{R}$ is a hyperring homomorphism from $K \to \mathsf{U}\nabla$.

non-archimedian = satisfies the ultra-metric triangle inequality

$$|a + b| \leq \max(a, b) \text{ for any } a, b \in K.$$

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Valuations are nothing but hyperring homomorphisms to $\mathbb{V}$!

Sign and phase

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The map

$$\mathrm{sign} : \mathbb{R} \to \{0, 1, -1\} : x \mapsto \begin{cases} \frac{x}{|x|}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

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The map
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is a hyperring homomorphism $\mathbb{C} \to \Phi$ and $\mathcal{T}\mathbb{C} \to \Phi$.

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Characteristic 1?

Traditional tropical geometry is a geometry of (degenerated) amoebas.

Hyperfields recover real and complex varieties in tropical geometry.

Dequantizataions

Litvinov-Maslov dequantization

For $h > 0$, consider a map $R_h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$$x \mapsto \begin{cases} x^{\frac{1}{h}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

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These are multiplicative homomorphisms, but they do not respect addition.

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$$= (a^{1/h} + b^{1/h})^h$$

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$R_h = (\mathbb{R}_{\geq 0}, +_h, \times)$ is a copy of semifield $(\mathbb{R}_{\geq 0}, +, \times)$ and
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P_h is a degeneration of $(\mathbb{R}_{\geq 0}, +, \times)$ to $(\mathbb{R}_{\geq 0}, \max, \times)$.

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P_h is a **dequantization** of $(\mathbb{R}_{\geq 0}, +, \times)$ to $(\mathbb{R}_{\geq 0}, \max, \times)$.

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These are multiplicative homomorphisms, but they do not respect $(a, b) \mapsto a \triangledown b$.

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The endpoints of segment $a \triangledown_h b$ tend
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Let $a \triangledown_0 b := a \curlyvee b$.

∇_h is a dequantization of ∇ to $\mathrm{U}\nabla$.

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For $h > 0$ consider a map $S_h\colon \mathbb{C} \to \mathbb{C}$

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These are multiplicative isomorphisms, but they do not respect the addition.

Dequantization $\mathbb{C}$ to $\mathcal{T}\mathbb{C}$

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Pull back the addition via S_h :

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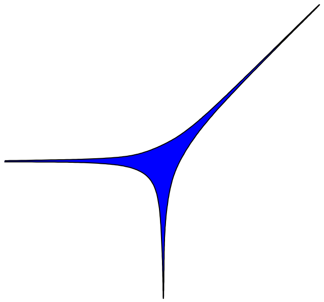

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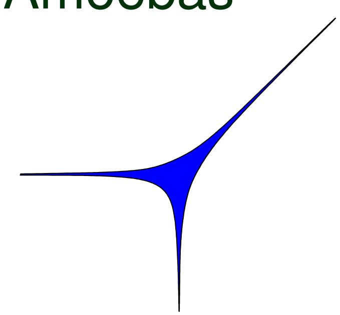

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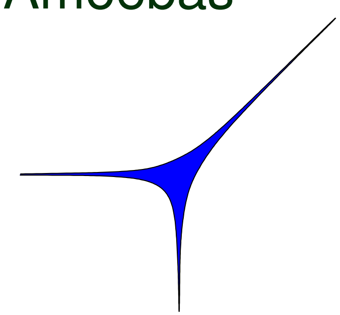

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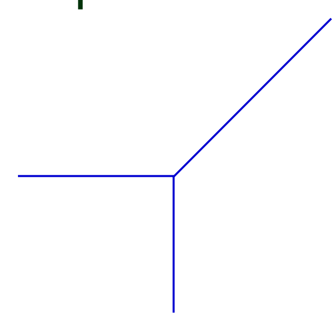

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Complex Algebraic Geometry — Complex Tropical Geometry

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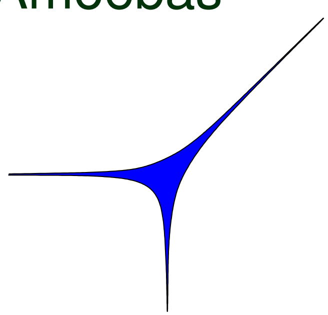

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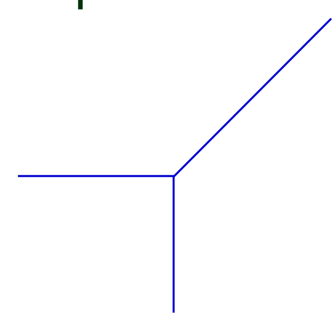

Geometries over Hyperfields

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Graphs and curves

In geometry over $\mathbb{T}$

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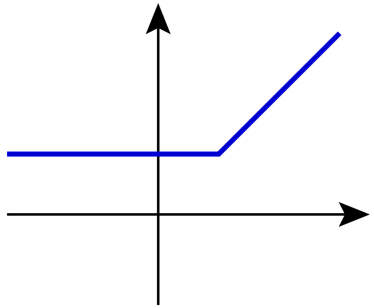

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Graphs and curves

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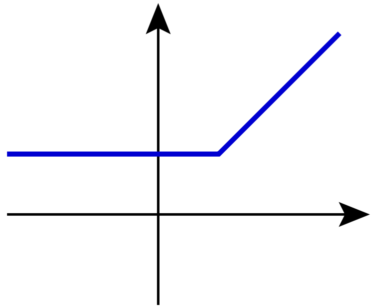

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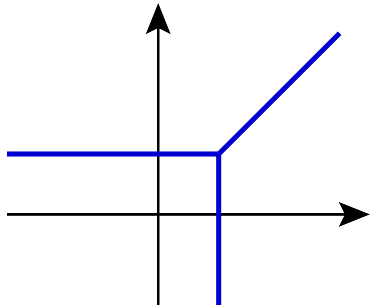

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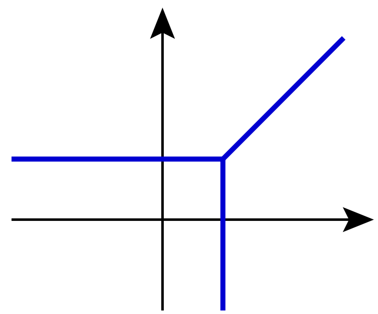

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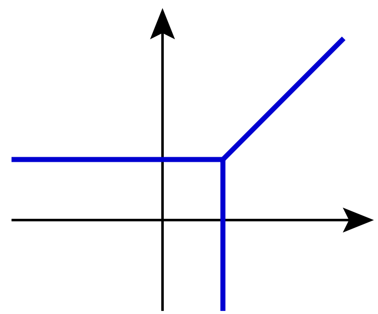

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Complex Tropical Geometry

Complex tropical line

$\{(x, y) \in \mathbb{C}^2 \mid 0 \in x \smile y \smile 1\}$

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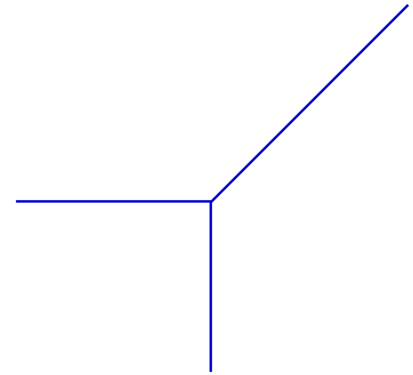

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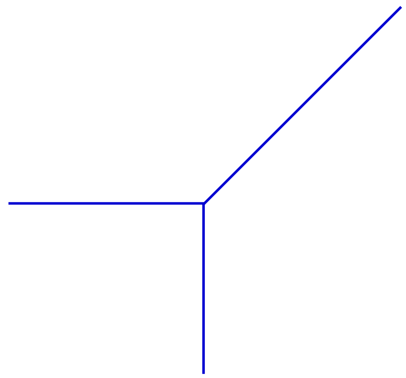

$\operatorname{Log}^{-1}(\text{a ray})$ is a holomorphic cylinder.

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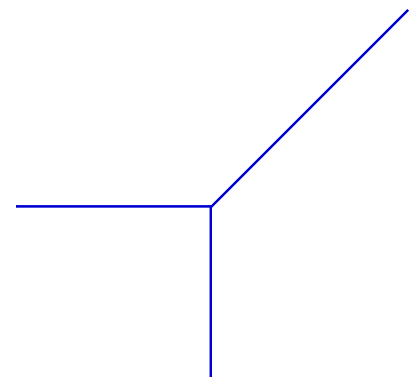
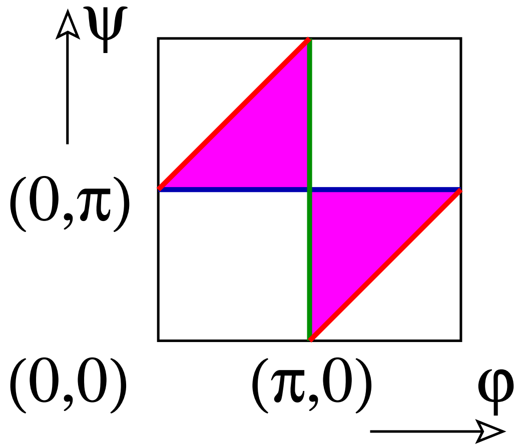


Log^{-1}(a ray) is a holomorphic cylinder.

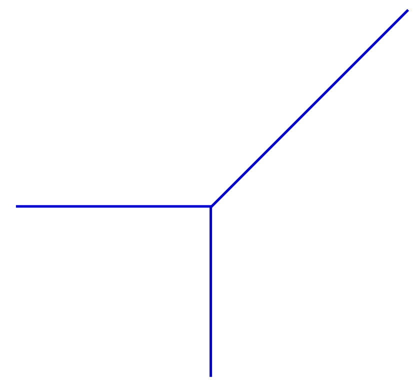
Log^{-1}(the central point) =



Complex tropical line

$\{(x,y) \in \mathbb{C}^2 \mid 0 \in x \smile y \smile 1\}$

The amoeba (the image under $\operatorname{Log} : (\mathbb{C} \setminus 0)^2 \to \mathbb{R}^2$) is the tropical line



Log^{-1}(a ray) is a holomorphic cylinder.

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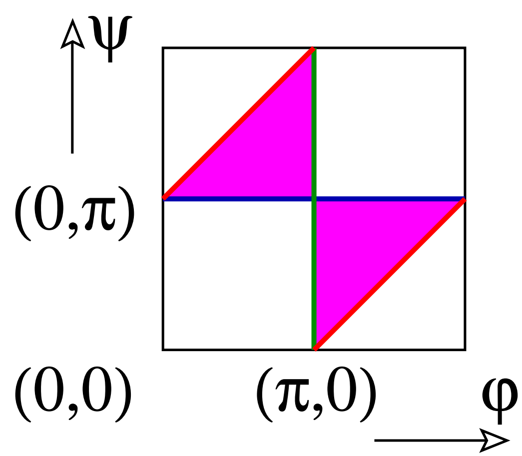

Overall a disk.

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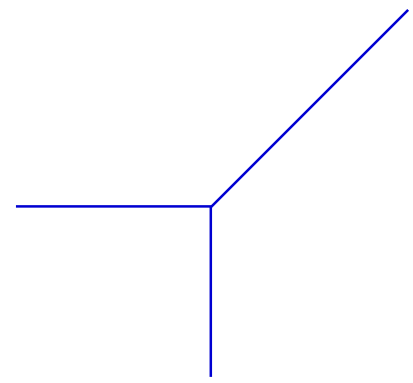

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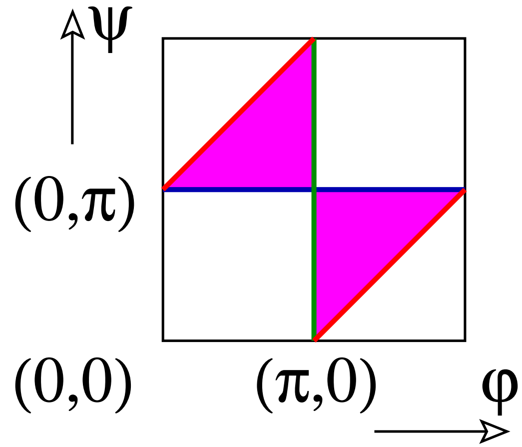

Overall a disk. A 2-manifold!

Complex tropical varieties

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Conjecture. If under the dequantization a non-singular complex varieties tends to a non-singular complex tropical variety, then the dequantization provides an isotopy between the varieties.

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Some polynomial functions

Is $x = x \smile_{\mathbb{R}} 1 \smile_{\mathbb{R}} -1$?

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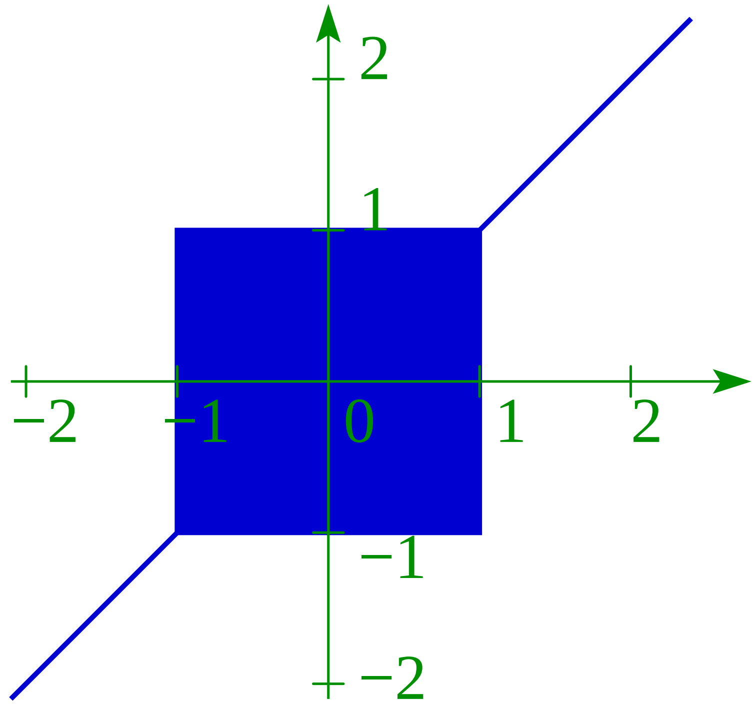

Graph of the function $y = x \smile_{\mathbb{R}} 1 \smile_{\mathbb{R}} -1$.

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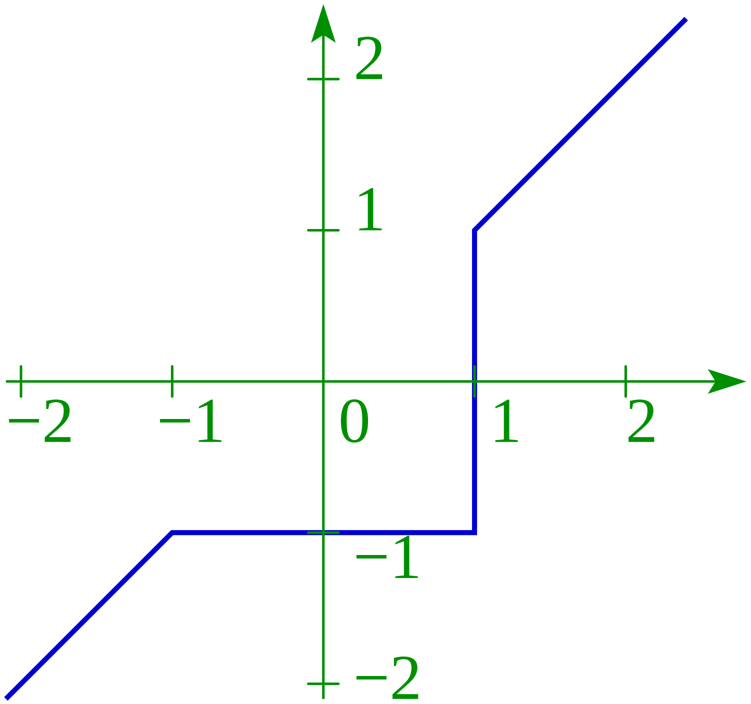

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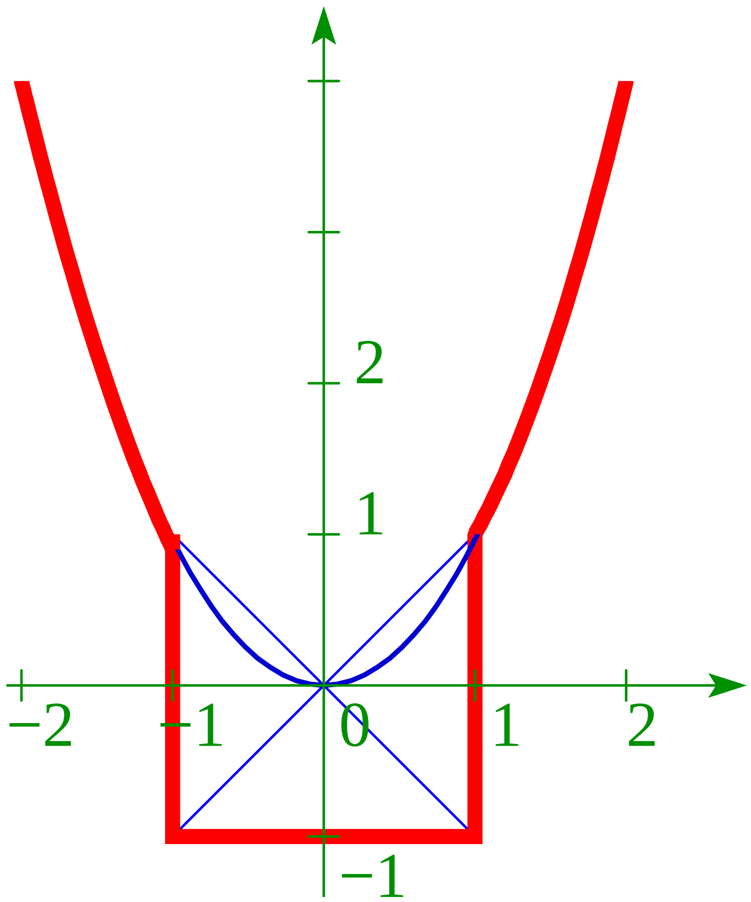

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Some hyperfields are double distributive, some are not.

In particular, $\mathcal{T}\mathbb{R}$, $\mathbf{K}$, $\mathbf{S}$ and $\mathbb{Y}$ are double distributive.

while $\mathcal{T}\mathbb{C}$, Φ and ∇ are not.

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$R[x_1, \ldots, x_n]$ is a hyperring with addition $\underline{\top}$ and usual multiplication.

Table of Contents

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