Khovanov homology of framed and signed chord diagrams.

Oleg Viro

December 2, 2006

Knots and links

Classical link diagrams

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That is a circle smoothly embedded into $\mathbb{R}^3$.

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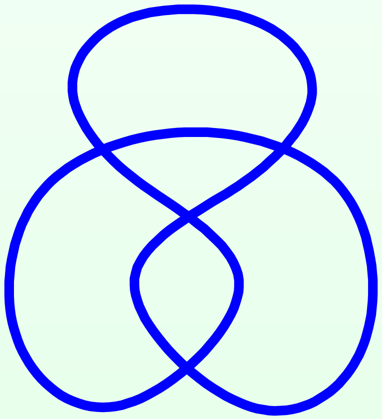

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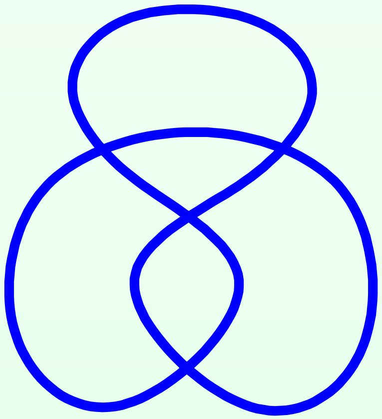

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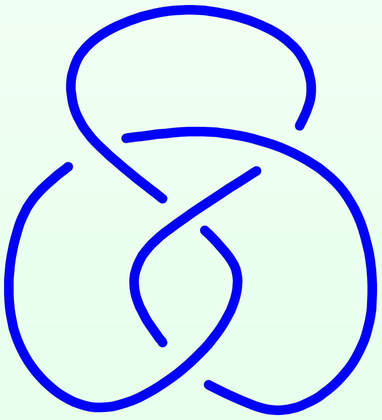

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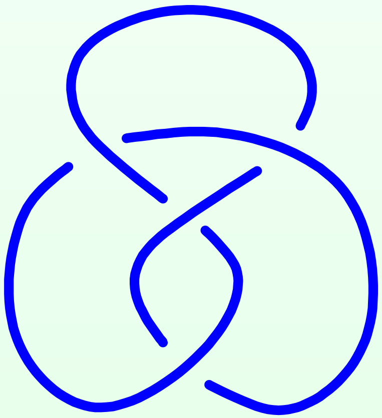

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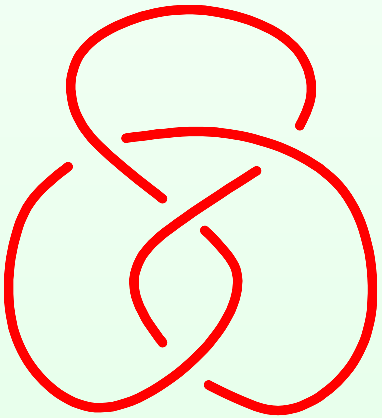

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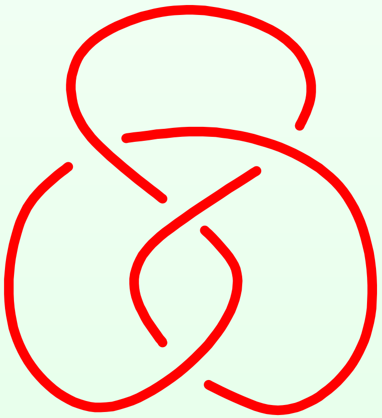

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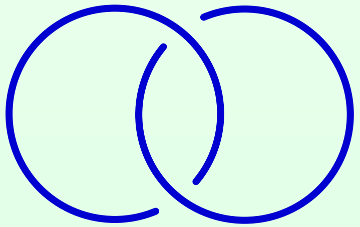

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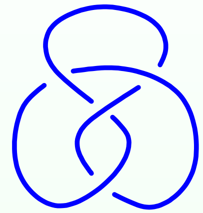
A knot diagram 

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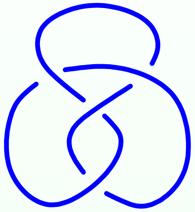
A knot diagram  is a 2D picture of knot.

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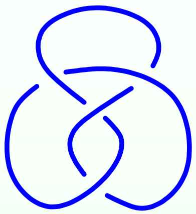
A knot diagram  is a 2D picture of knot.

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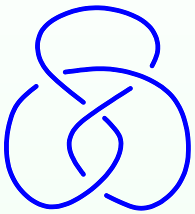
A knot diagram  is a 2D picture of knot.

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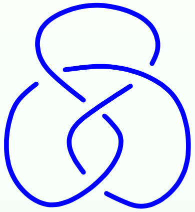
A knot diagram  is a 2D picture of knot.

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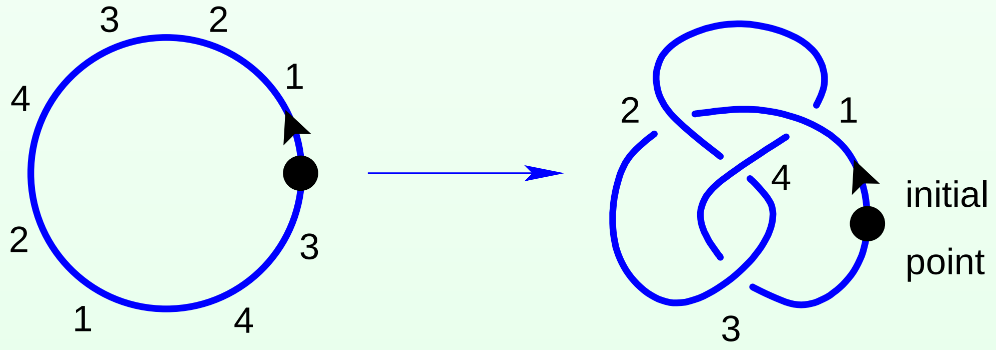
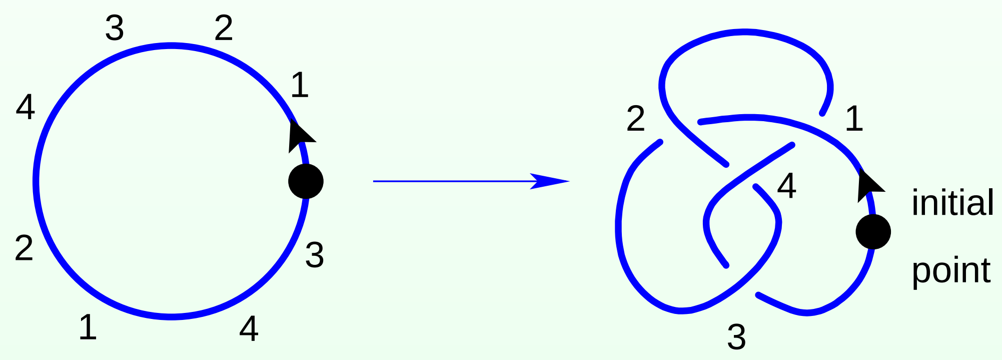


Gauss diagram

Decorate the source:



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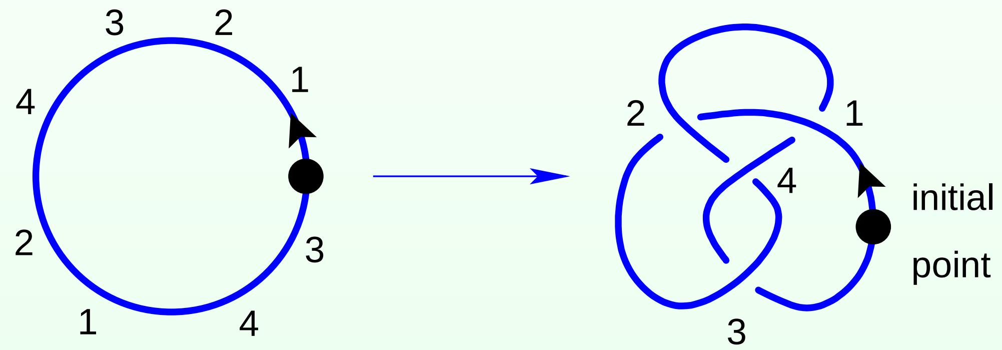

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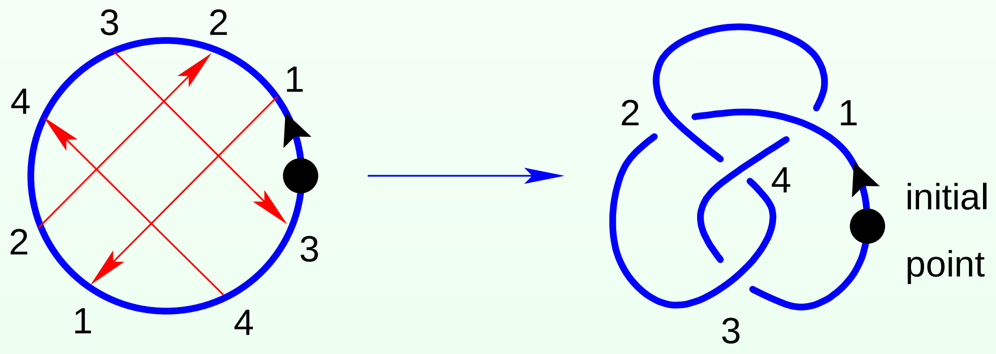

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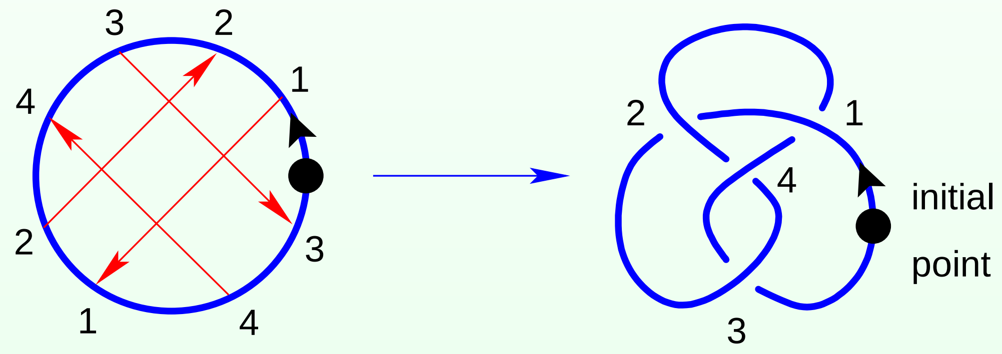

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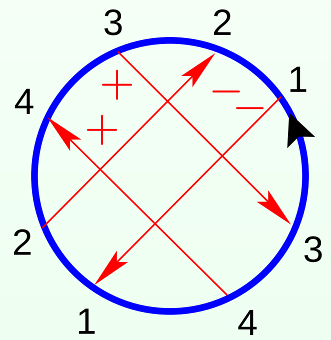

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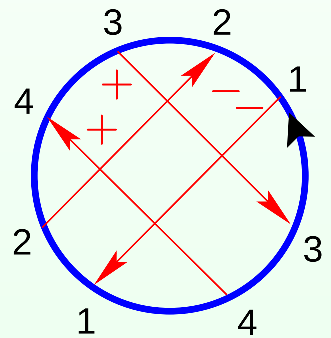

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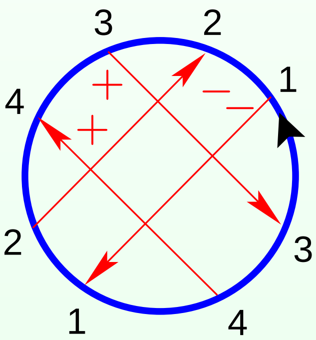
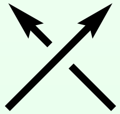


Signs: positive



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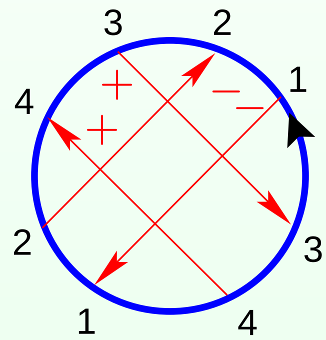
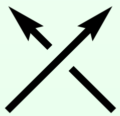
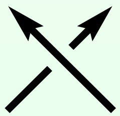


Signs: positive  , negative  .

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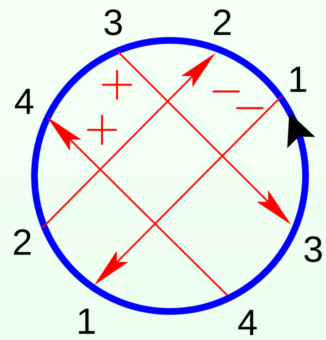
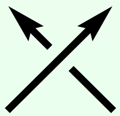
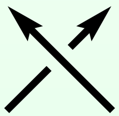


Signs: positive  , negative  . The result

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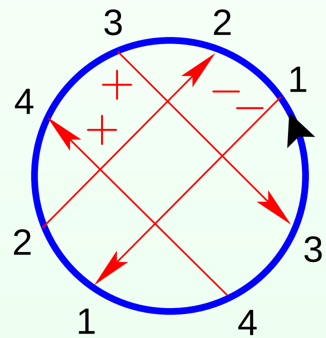
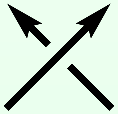
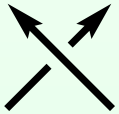
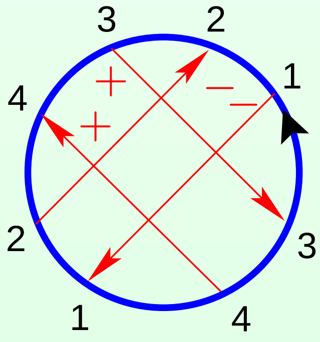


Signs: positive  , negative  . The result,



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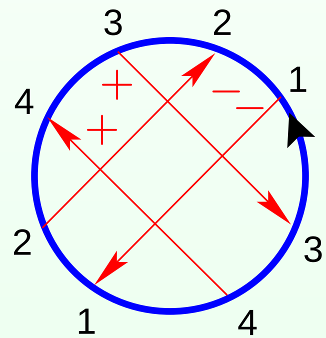
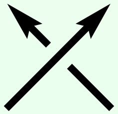
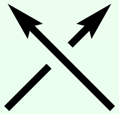
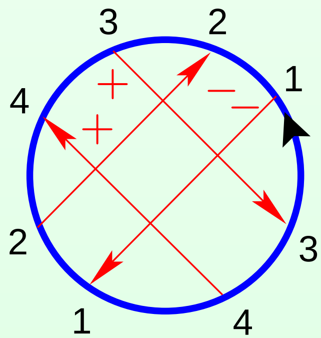


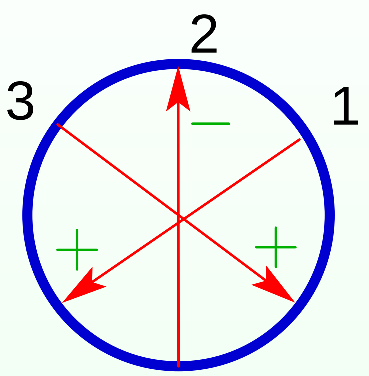
Signs: positive  , negative  . The result,



, is called a *Gauss diagram* of the knot.

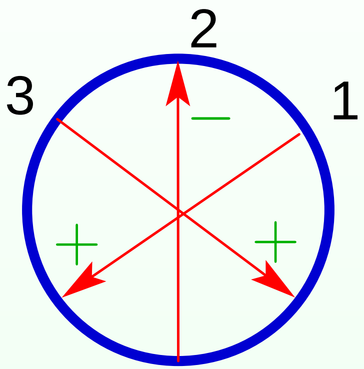
Reconstruction of knot

Take any such diagram, say,



and try to reconstruct the knot.

Reconstruction of knot



Start with crossings:

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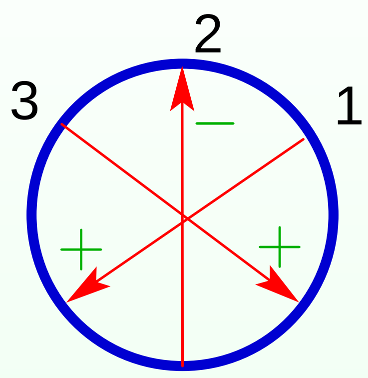

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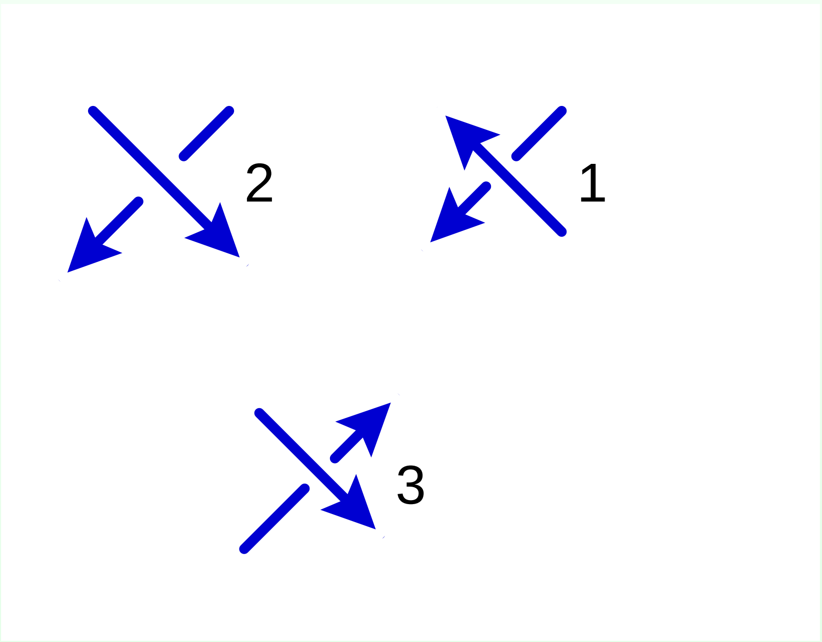

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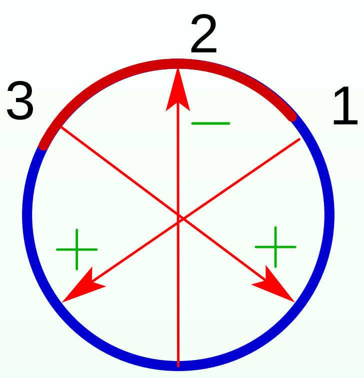
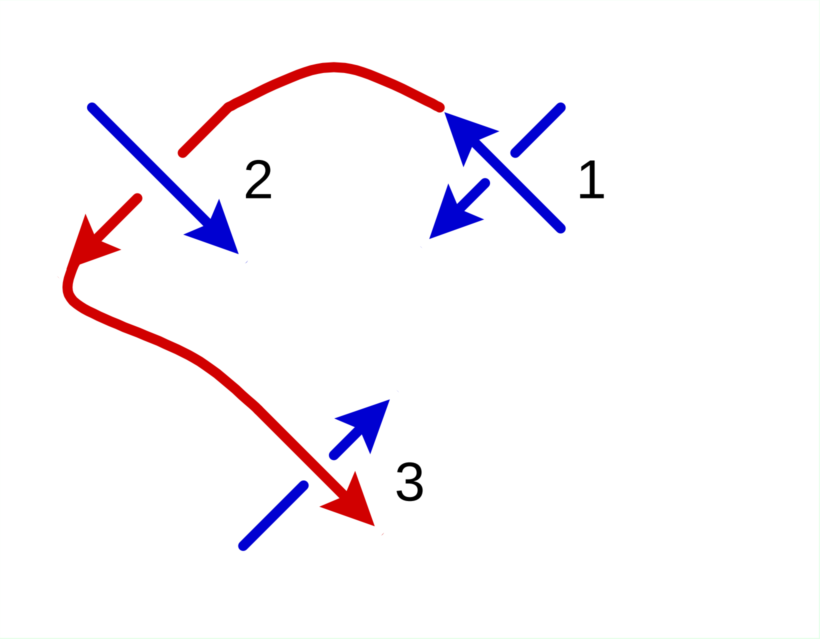


Connect them step by step:



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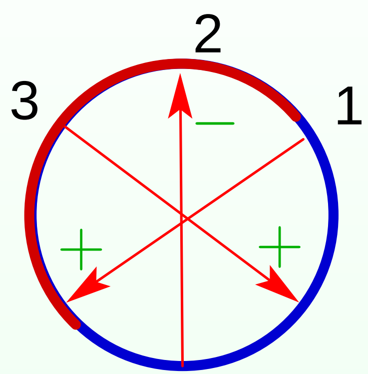

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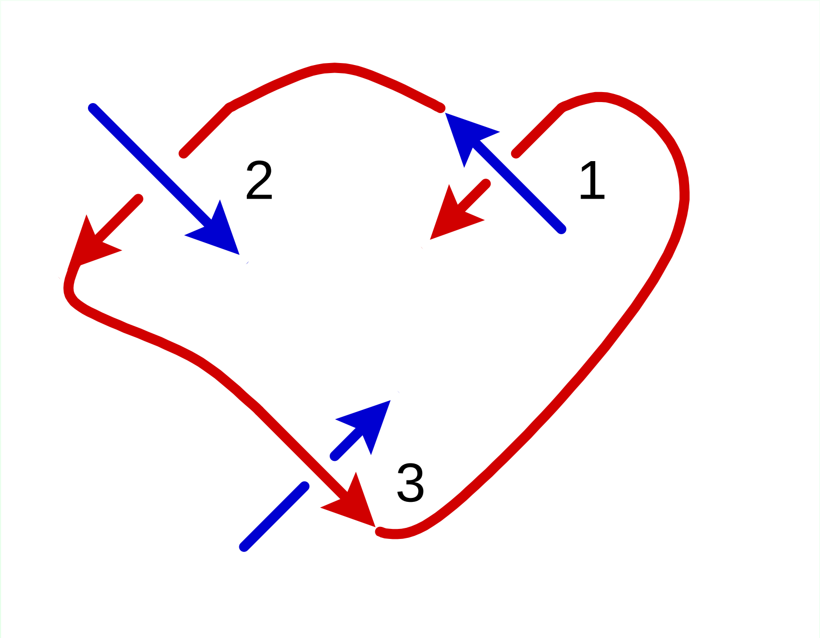

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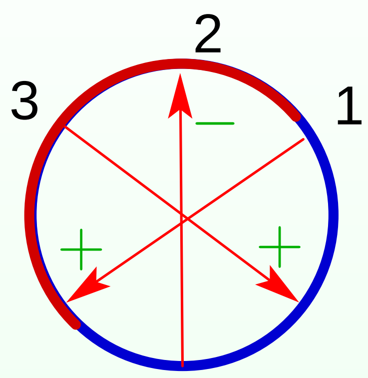
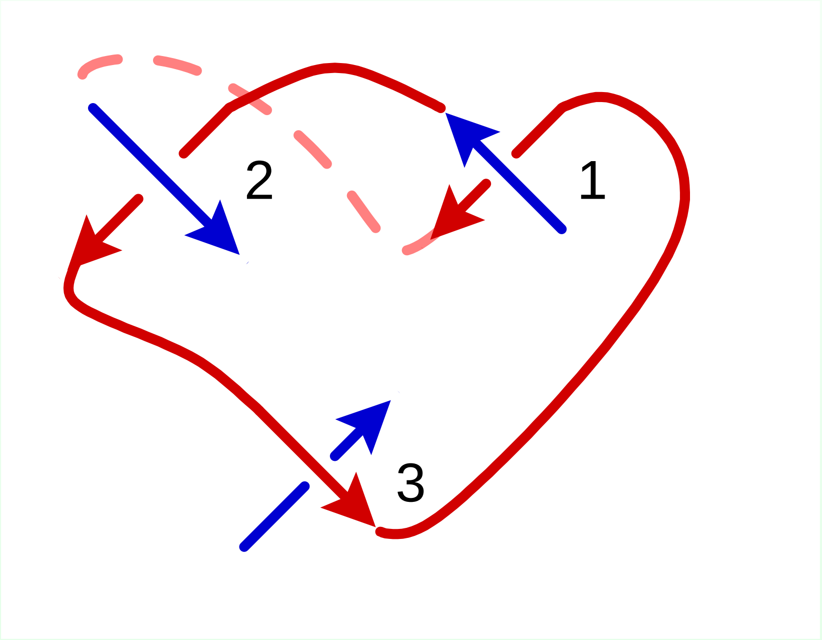


The next step does not work!



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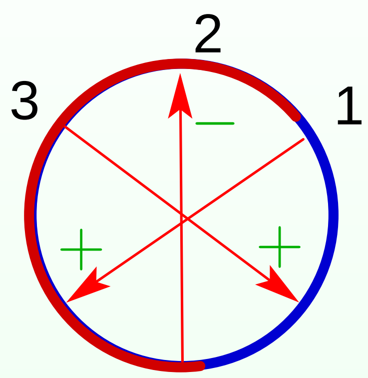
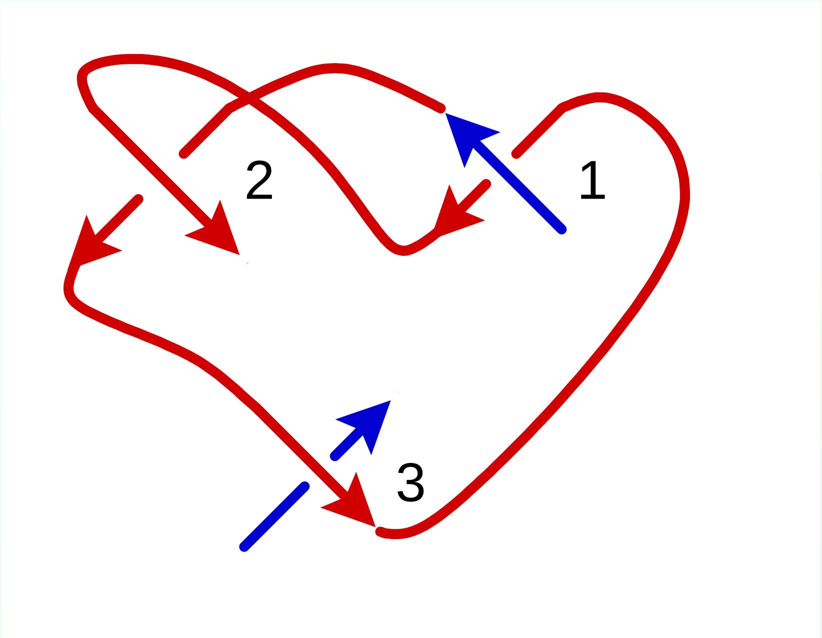


But let us continue!



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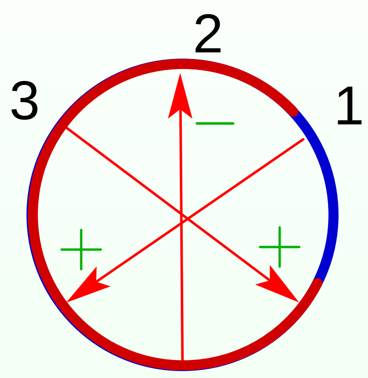
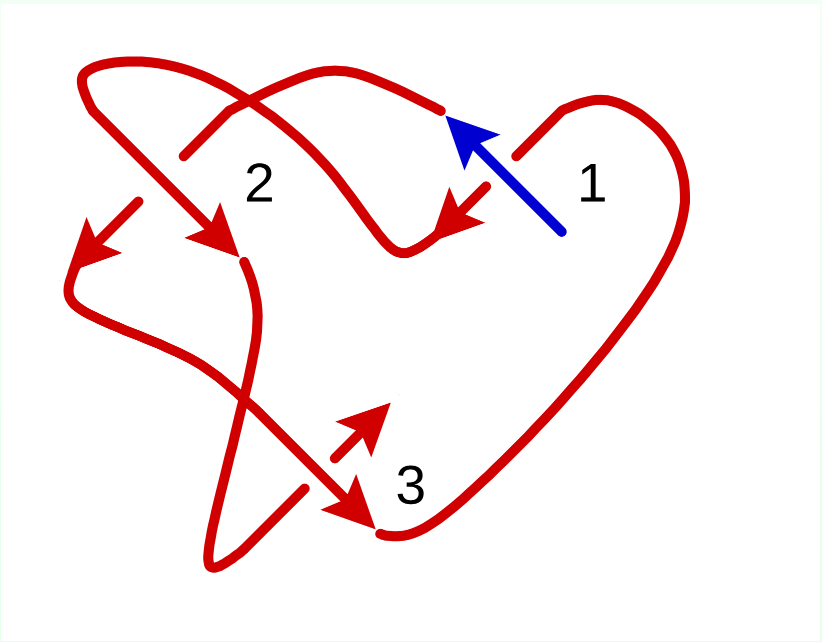


Yet another obstruction!



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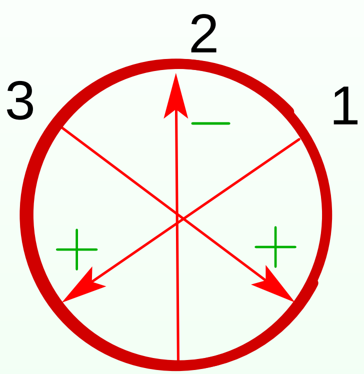
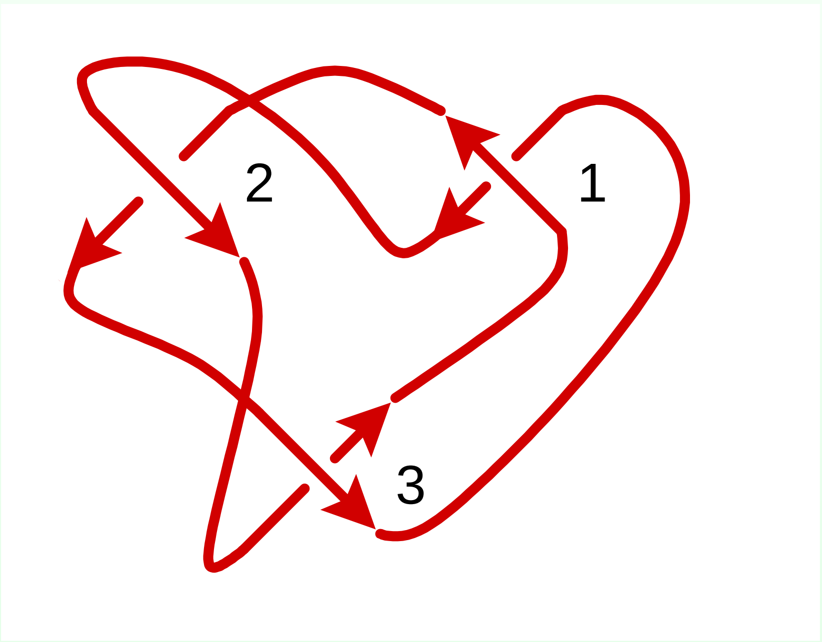


We did it!



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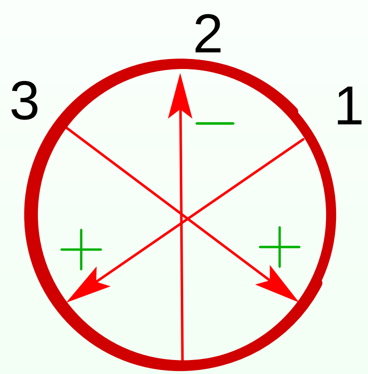
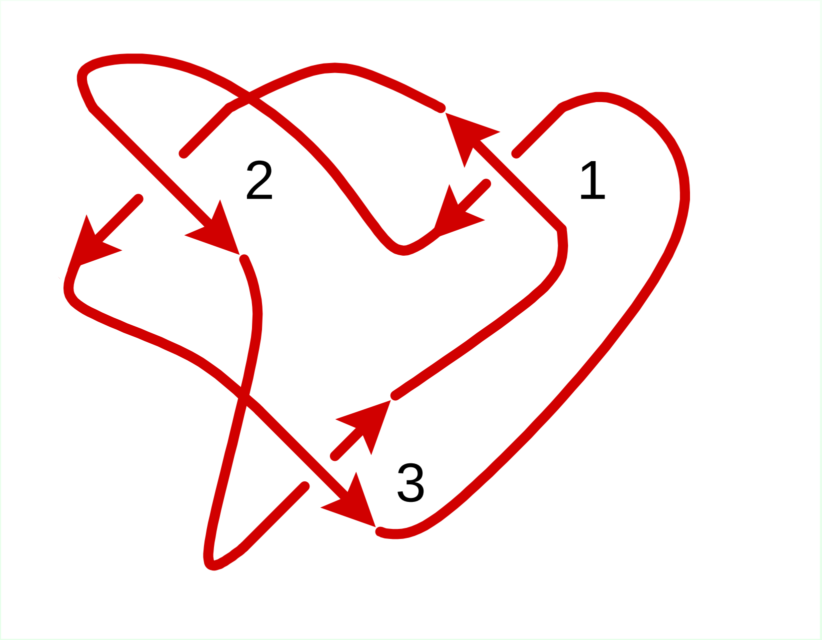


We did it! But what is the result?



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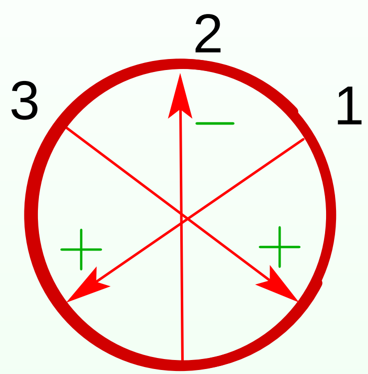

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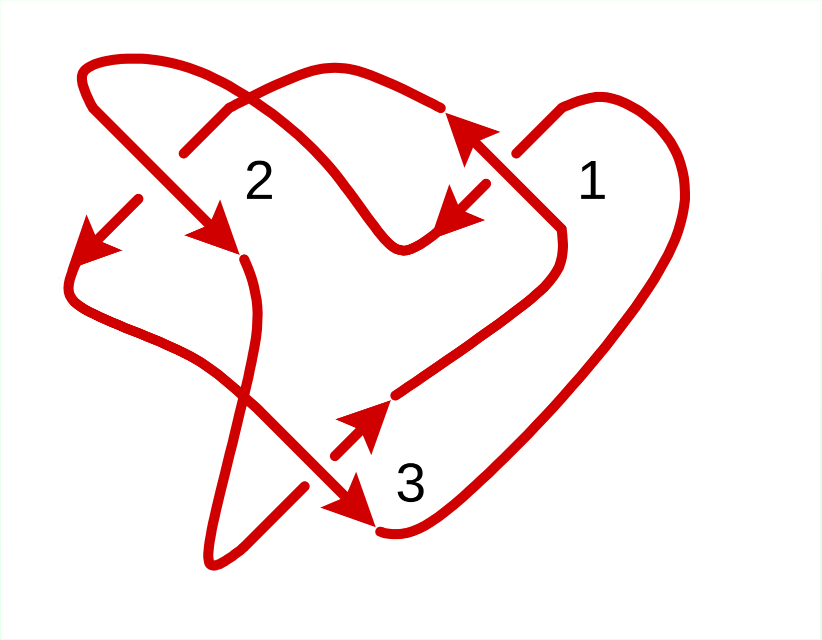

The result is called a virtual knot diagram.

Virtual links

Virtual knot diagrams

A virtual knot diagram has crossings of 2 types:

Virtual knot diagrams

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classical

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A virtual knot diagram has crossings of 2 types:
classical or real

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classical or real   decorated like in a knot diagram

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Who can help to get rid of virtual crossings?

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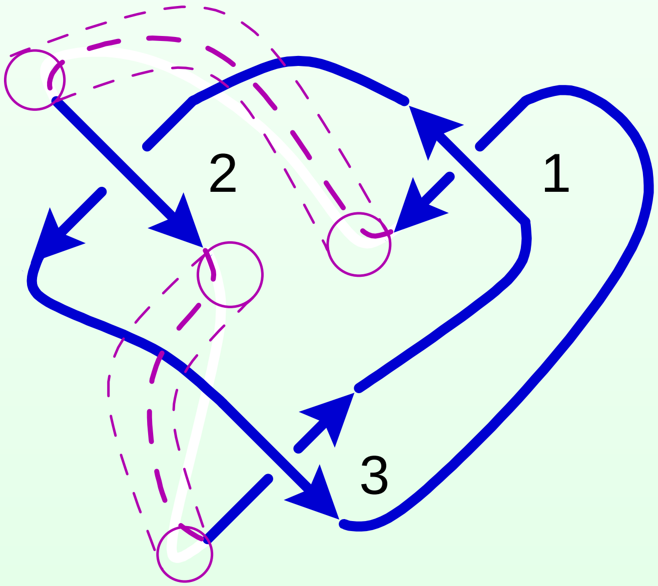

Diagram on a surface

A knot diagram drawn on orientable surface S

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A knot diagram drawn on orientable surface S, instead of the plane

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The surfaces is not unique: one can add handles.

Moves

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What happens to a link diagram, when the link moves?

Moves

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Link diagram moves, too.

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Reidemeister moves:

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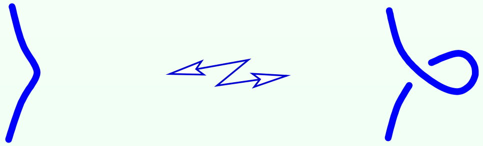
(R1): 

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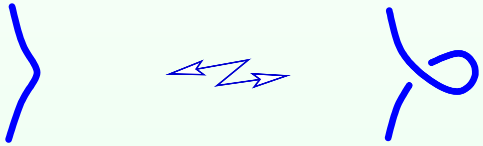
Reidemeister moves:

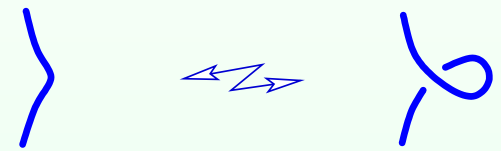
(R1): 

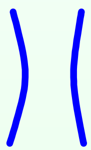
(R2):

Moves

What happens to a link diagram, when the link moves?
Link diagram moves, too.

Reidemeister moves:

(R1): 

(R2): 

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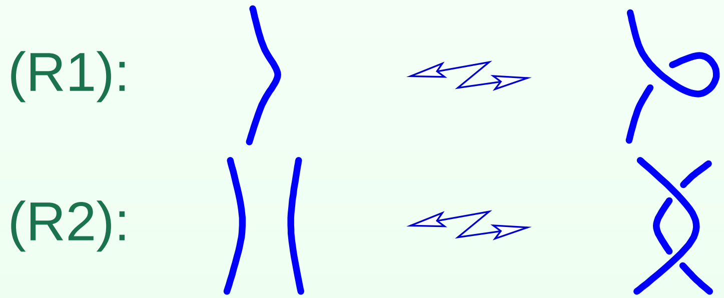
Reidemeister moves:



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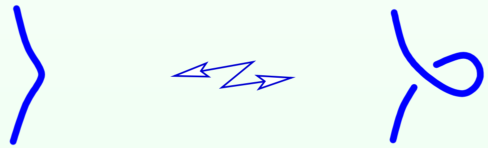
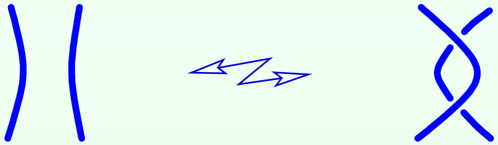
Reidemeister moves:

(R1): 

(R2): 

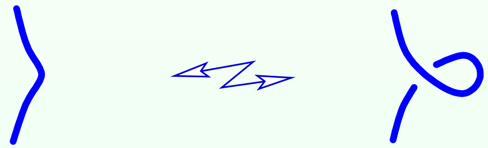
(R3):

Moves

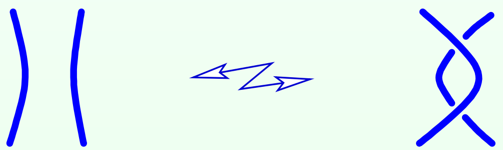
What happens to a link diagram, when the link moves?
Link diagram moves, too.

Reidemeister moves:

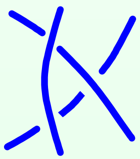
(R1):



(R2):



(R3):



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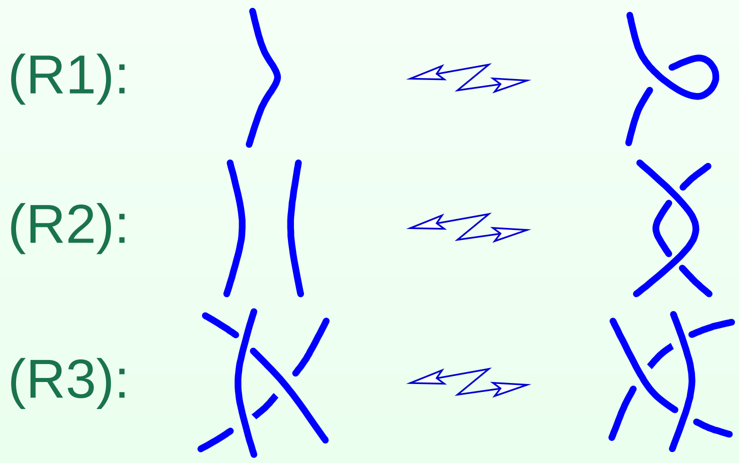
Reidemeister moves:



Moves of virtual link diagram

A virtual link diagram

(i.e., a plane projection of a link diagram on a surface)

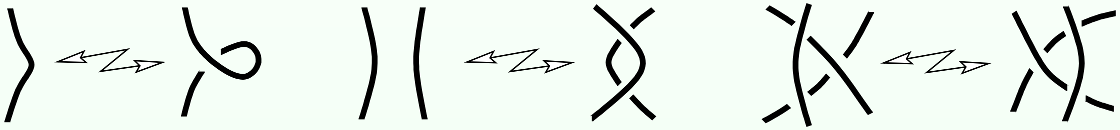
Moves of virtual link diagram

A virtual link diagram moves like this:

Moves of virtual link diagram

A virtual link diagram moves like this:

Reidemeister moves:

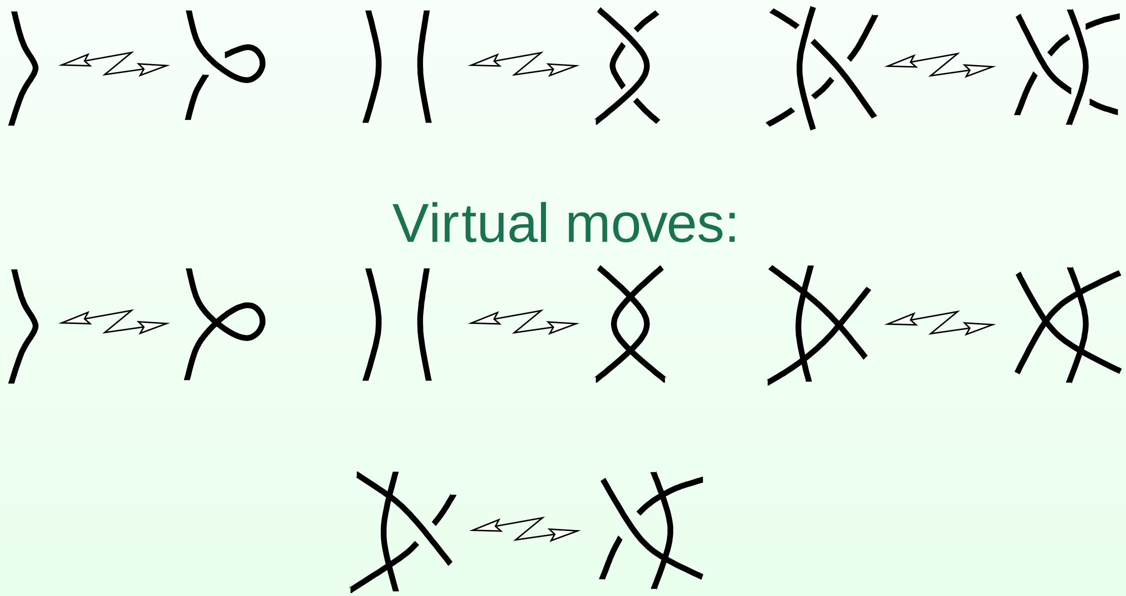


Moves of virtual link diagram

A virtual link diagram moves like this:

Reidemeister moves:

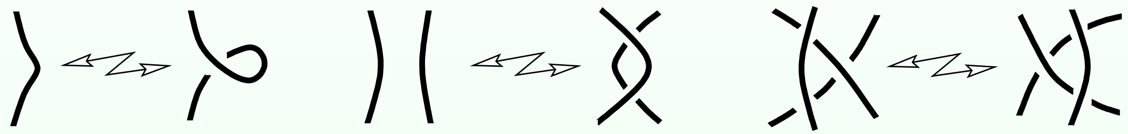
Virtual moves:



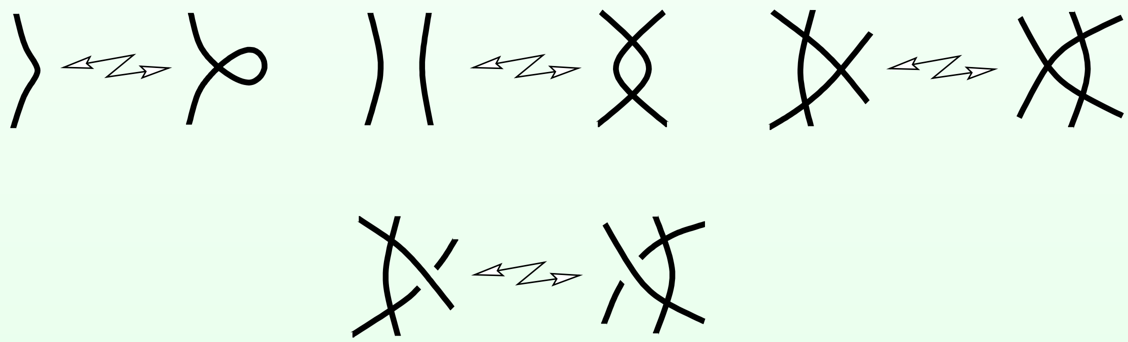
Moves of virtual link diagram

A virtual link diagram moves like this:

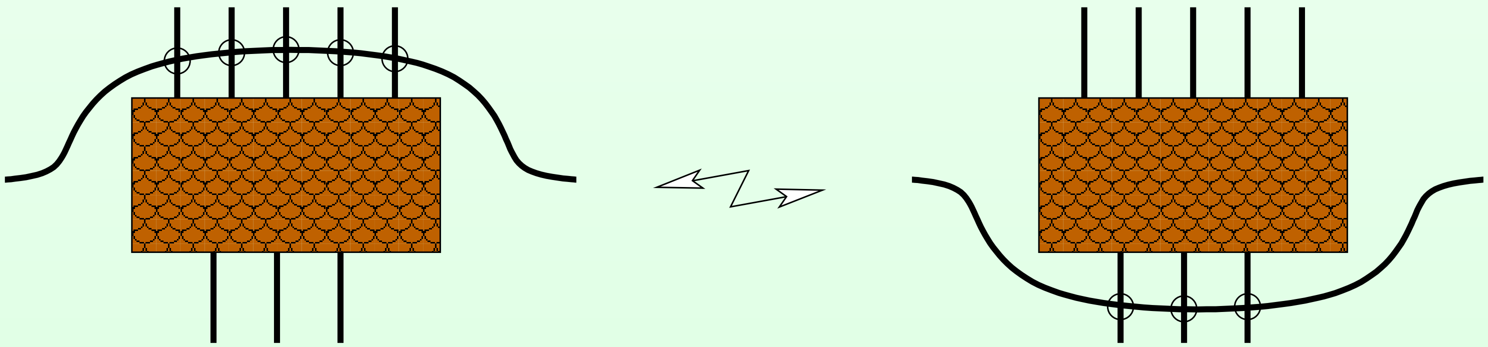
Reidemeister moves:



Virtual moves:



All virtual moves can be replaced by detour moves:



Moves of Gauss diagrams

Gauss diagrams has nothing to do with virtual crossings!

Moves of Gauss diagrams

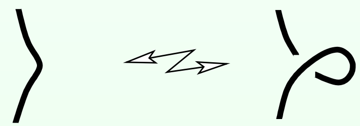
Gauss diagrams has nothing to do with virtual crossings!
They do not change under virtual moves.

Moves of Gauss diagrams

Reidemeister moves acts on Gauss diagram:

Moves of Gauss diagrams

Reidemeister moves acts on Gauss diagram:

Move's name	Reidemeister move	Its action on Gauss diagram
Positive first move		
Negative first move		

Moves of Gauss diagrams

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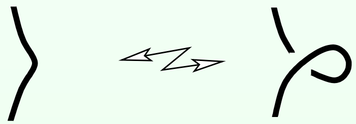
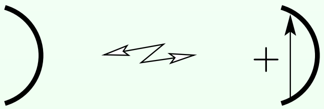
Positive first move		
Negative first move		

Moves of Gauss diagrams

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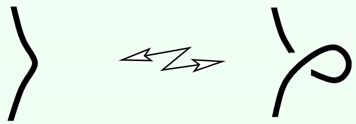
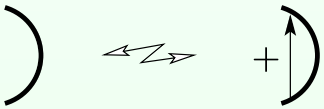
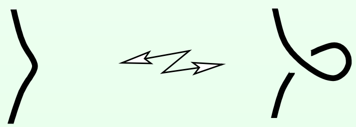
Move's name	Reidemeister move	Its action on Gauss diagram
Positive first move		
Negative first move		

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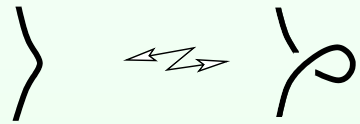
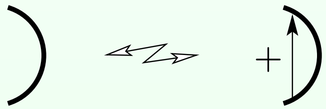
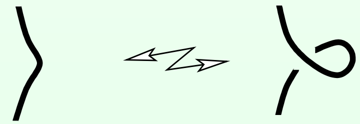
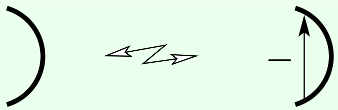
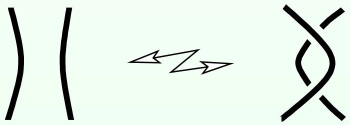
Positive first move		
Negative first move		

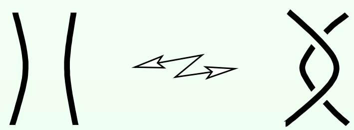
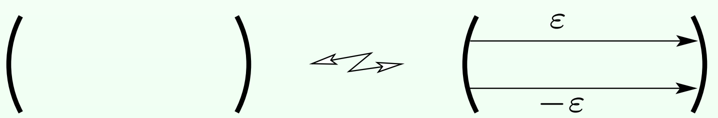
Moves of Gauss diagrams

Reidemeister moves acts on Gauss diagram:

Move's name	Reidemeister move	Its action on Gauss diagram
Second move		
Third move		

Moves of Gauss diagrams

Reidemeister moves acts on Gauss diagram:

Move's name	Reidemeister move	Its action on Gauss diagram
Second move		
Third move		

Moves of Gauss diagrams

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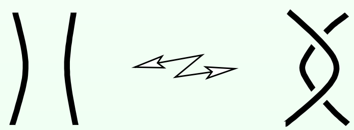
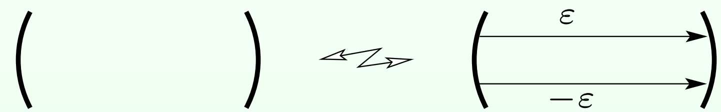
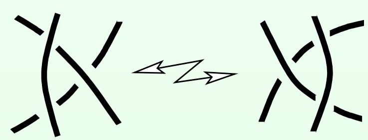
Second move		
Third move		

Moves of Gauss diagrams

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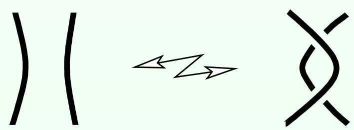
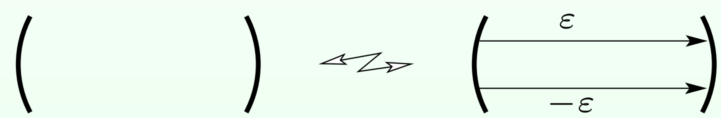
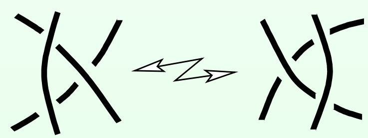
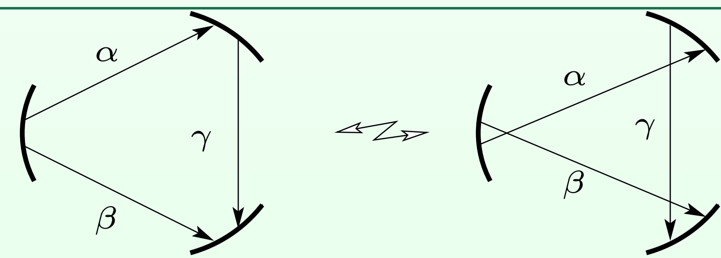
Second move		
Third move		

Combinatorial incarnation of knot theory

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Classical Links $\longrightarrow$ Link diagrams

Combinatorial incarnation of knot theory

Classical Links	$\longrightarrow$	Link diagrams
Isotopies	$\longrightarrow$	Reidemeister moves

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Combinatorial incarnations of virtual knot theory

Combinatorial incarnation of knot theory

Classical Links	$\longrightarrow$	Link diagrams
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Combinatorial incarnations of virtual knot theory

Gauss Diagrams	$\longleftarrow$	Virtual Links (?)	$\longrightarrow$	Virtual Link Diagrams
Reidemeister Moves	$\longleftarrow$	Virtual Istopies (?)	$\longrightarrow$	Reidemeister and Detour moves

Topological meaning of virtual knot theory

Third incarnation of virtual knot theory
is provided by Kuperberg's theorem.

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Implies that virtual links generalize classical ones.

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Bridges combinatorics

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Bridges combinatorics (= 1D topology)

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Third incarnation of virtual knot theory is provided by Kuperberg's theorem.

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Implies that virtual links generalize classical ones.

Bridges combinatorics with (3D-) topology.

Isotopy problem

Isotopy Problem:

Isotopy problem

Isotopy Problem: Are given two classical links isotopic?

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Combinatorial reformulation:

Isotopy problem

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Can given two Gauss diagrams be related by moves?

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Invariants needed!

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Invariants needed!

The most classical link invariant is the *link group*.

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The most classical link invariant is the *link group*,

the fundamental group of the link complement $\mathbb{R}^3 \setminus$ link .

Isotopy problem

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Invariants needed!

The most classical link invariant is the *link group*.

It was generalized, even in two ways!

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It was generalized: upper and lower!

In terms of links in a thickened surface this is the fundamental group of the complement, but with one of two sides of the boundary contracted to a point.

Kauffman bracket is more practical and elementary invariant.

Kauffman bracket

Kauffman bracket

$$\langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}]$$

Kauffman bracket

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(a Laurent polynomial in A with integer coefficients).

Kauffman bracket

$$\langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}]$$

$\langle \text{unknot} \rangle =$

Kauffman bracket

$$\langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}]$$

$\langle \text{unknot} \rangle =$ $\qquad$ $\langle \bigcirc \rangle =$

Kauffman bracket

$$\langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}]$$

$$\langle \text{unknot} \rangle = \qquad \langle \bigcirc \rangle = -A^2 - A^{-2}$$

Kauffman bracket

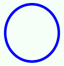
$$\langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}]$$

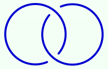
$$\langle \text{unknot} \rangle = \qquad \langle \bigcirc \rangle = -A^2 - A^{-2}$$

$$\langle \text{Hopf link} \rangle =$$

Kauffman bracket

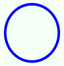
$$\langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}]$$

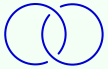
$\langle \text{unknot} \rangle =$ $\langle$  $\rangle = -A^2 - A^{-2}$

$\langle \text{Hopf link} \rangle =$ $\langle$  $\rangle =$

Kauffman bracket

$$\langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}]$$

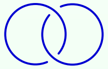
$\langle \text{unknot} \rangle = \langle$  $\rangle = -A^2 - A^{-2}$

$\langle \text{Hopf link} \rangle = \langle$  $\rangle = A^6 + A^2 + A^{-2} + A^{-6}$

Kauffman bracket

$$\langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}]$$

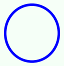
$\langle \text{unknot} \rangle =$ $\langle$  $\rangle = -A^2 - A^{-2}$

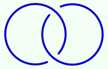
$\langle \text{Hopf link} \rangle =$ $\langle$  $\rangle = A^6 + A^2 + A^{-2} + A^{-6}$

$\langle \text{empty link} \rangle =$ $\langle \quad \rangle =$

Kauffman bracket

$$\langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}]$$

$\langle \text{unknot} \rangle =$ $\langle$  $\rangle = -A^2 - A^{-2}$

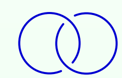
$\langle \text{Hopf link} \rangle =$ $\langle$  $\rangle = A^6 + A^2 + A^{-2} + A^{-6}$

$\langle \text{empty link} \rangle =$ $\langle\ \rangle = 1$

Kauffman bracket

$$\langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}]$$

$\langle \text{unknot} \rangle =$ ⟨  ⟩ $= -A^2 - A^{-2}$

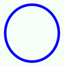
$\langle \text{Hopf link} \rangle =$ ⟨  ⟩ $= A^6 + A^2 + A^{-2} + A^{-6}$

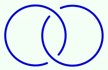
$\langle \text{empty link} \rangle = \langle \ \rangle = 1$

$\langle \text{trefoil} \rangle =$

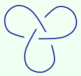
Kauffman bracket

$$\langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}]$$

$\langle \text{unknot} \rangle =$ ⟨  ⟩ $= -A^2 - A^{-2}$

$\langle \text{Hopf link} \rangle =$ ⟨  ⟩ $= A^6 + A^2 + A^{-2} + A^{-6}$

$\langle \text{empty link} \rangle = \langle \quad \rangle = 1$

$\langle \text{trefoil} \rangle =$ ⟨  ⟩ $=$

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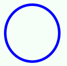
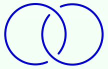
$\langle \text{unknot} \rangle =$ $\langle$  $\rangle = -A^2 - A^{-2}$

$\langle \text{Hopf link} \rangle =$ $\langle$  $\rangle = A^6 + A^2 + A^{-2} + A^{-6}$

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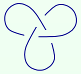
$\langle \text{trefoil} \rangle =$ $\langle$  $\rangle = A^7 + A^3 + A^{-1} - A^{-9}$

Kauffman bracket

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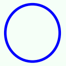
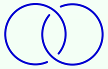
$\langle \text{unknot} \rangle =$ $\langle$  $\rangle = -A^2 - A^{-2}$

$\langle \text{Hopf link} \rangle =$ $\langle$  $\rangle = A^6 + A^2 + A^{-2} + A^{-6}$

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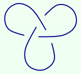
$\langle \text{trefoil} \rangle =$ $\langle$  $\rangle = A^7 + A^3 + A^{-1} - A^{-9}$

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Kauffman bracket

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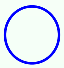
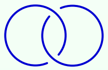
$\langle$unknot$\rangle =$ $\langle$  $\rangle = -A^2 - A^{-2}$

$\langle$Hopf link$\rangle =$ $\langle$  $\rangle = A^6 + A^2 + A^{-2} + A^{-6}$

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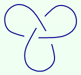
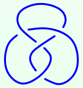
$\langle$trefoil$\rangle =$ $\langle$  $\rangle = A^7 + A^3 + A^{-1} - A^{-9}$

$\langle$figure-eight knot$\rangle =$ $\langle$  $\rangle =$

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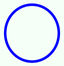
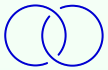
$\langle \text{unknot} \rangle =$ $\langle$  $\rangle = -A^2 - A^{-2}$

$\langle \text{Hopf link} \rangle =$ $\langle$  $\rangle = A^6 + A^2 + A^{-2} + A^{-6}$

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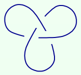
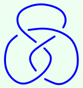
$\langle \text{trefoil} \rangle =$ $\langle$  $\rangle = A^7 + A^3 + A^{-1} - A^{-9}$

$\langle \text{figure-eight knot} \rangle =$ $\langle$  $\rangle = -A^{10} - A^{-10}$

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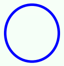
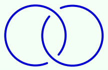
$\langle \text{unknot} \rangle =$ $\langle$  $\rangle = -A^2 - A^{-2}$

$\langle \text{Hopf link} \rangle =$ $\langle$  $\rangle = A^6 + A^2 + A^{-2} + A^{-6}$

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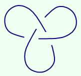
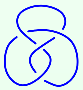
$\langle \text{trefoil} \rangle =$ $\langle$  $\rangle = A^7 + A^3 + A^{-1} - A^{-9}$

$\langle \text{figure-eight knot} \rangle =$ $\langle$  $\rangle = -A^{10} - A^{-10}$

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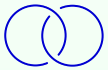
$\langle$Hopf link$\rangle =$ $\langle$  $\rangle = A^6 + A^2 + A^{-2} + A^{-6}$

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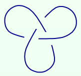
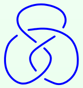
$\langle$trefoil$\rangle =$ $\langle$  $\rangle = A^7 + A^3 + A^{-1} - A^{-9}$

$\langle$figure-eight knot$\rangle =$ $\langle$  $\rangle = -A^{10} - A^{-10}$

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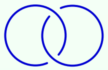
$\langle$Hopf link$\rangle =$ $\langle$  $\rangle = A^6 + A^2 + A^{-2} + A^{-6}$

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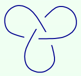
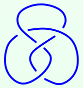
$\langle$trefoil$\rangle =$ $\langle$  $\rangle = A^7 + A^3 + A^{-1} - A^{-9}$

$\langle$figure-eight knot$\rangle =$ $\langle$  $\rangle = -A^{10} - A^{-10}$

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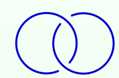
$\langle \text{Hopf link} \rangle = \langle$  $\rangle = A^6 + A^2 + A^{-2} + A^{-6}$

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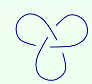
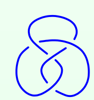
$\langle \text{trefoil} \rangle = \langle$  $\rangle = A^7 + A^3 + A^{-1} - A^{-9}$

$\langle \text{figure-eight knot} \rangle = \langle$  $\rangle = -A^{10} - A^{-10}$

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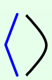
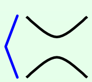
3. $\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$ (*Kauffman Skein Relation*) .

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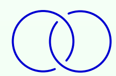
$\langle \text{Hopf link} \rangle = \langle$  $\rangle = A^6 + A^2 + A^{-2} + A^{-6}$

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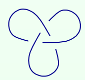
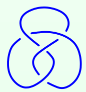
$\langle \text{trefoil} \rangle = \langle$  $\rangle = A^7 + A^3 + A^{-1} - A^{-9}$

$\langle \text{figure-eight knot} \rangle = \langle$  $\rangle = -A^{10} - A^{-10}$

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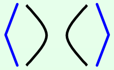
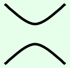
3. $\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$ (*Kauffman Skein Relation*) .

Uniqueness is obvious.

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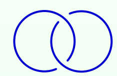
$\langle \text{Hopf link} \rangle = \langle$  $\rangle = A^6 + A^2 + A^{-2} + A^{-6}$

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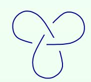
$\langle \text{trefoil} \rangle = \langle$  $\rangle = A^7 + A^3 + A^{-1} - A^{-9}$

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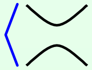
3. $\langle$  $\rangle = A\langle)(\rangle + A^{-1}\langle$  $\rangle$ (*Kauffman Skein Relation*) .

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Invariant under R2 and R3, under R1 multiplies by $-A^{\pm 3}$.

Kauffman state sum. I

A *state* of diagram is a distribution of *markers* over all crossings.

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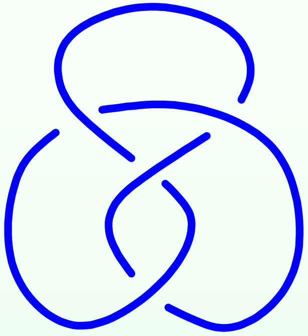
Knot diagram:



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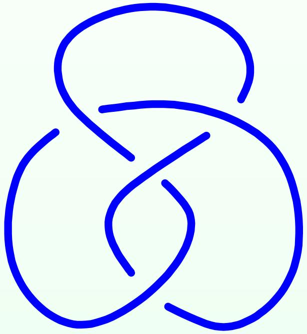
Knot diagram:  and its states:

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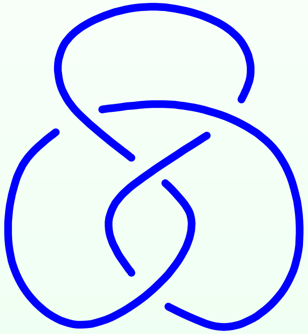
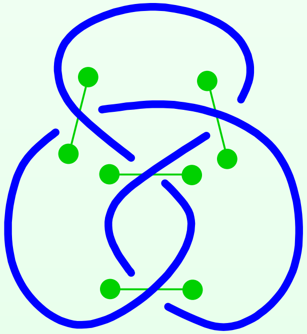
Knot diagram:  and its states:

 ,

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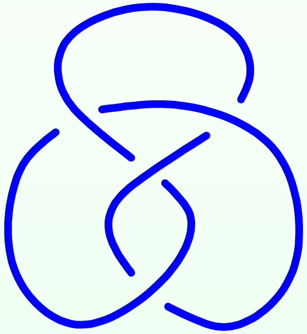
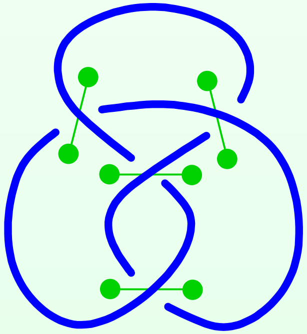
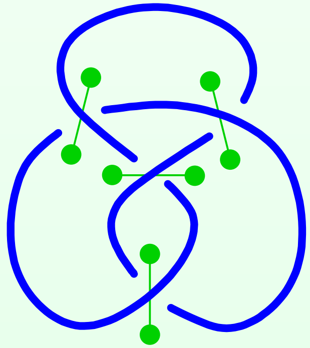
Knot diagram:  and its states:

 ,  ,

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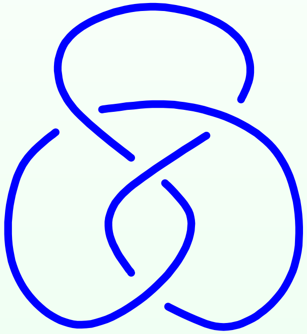
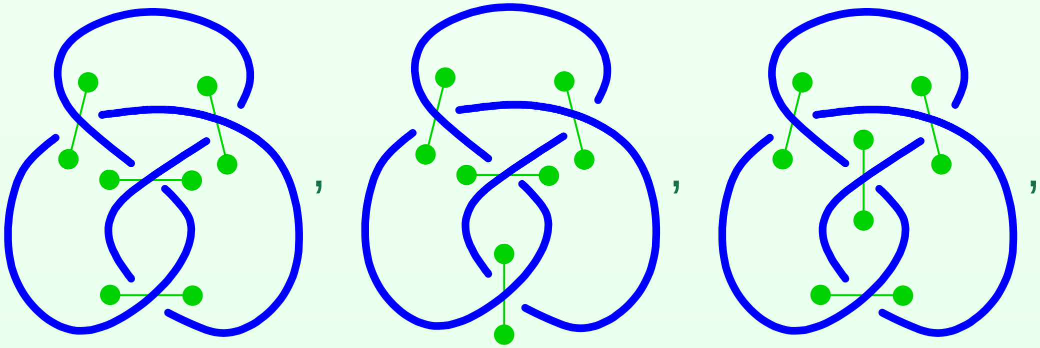
Knot diagram:  and its states:



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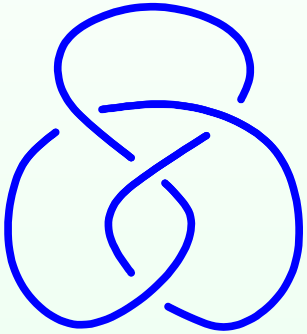
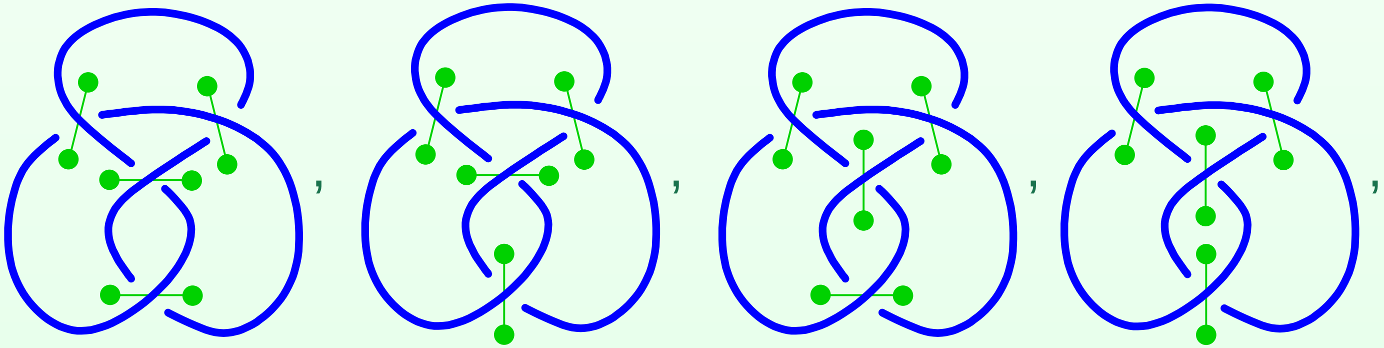
Knot diagram:  and its states:



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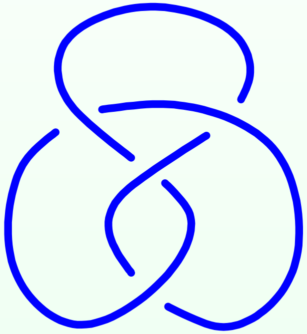
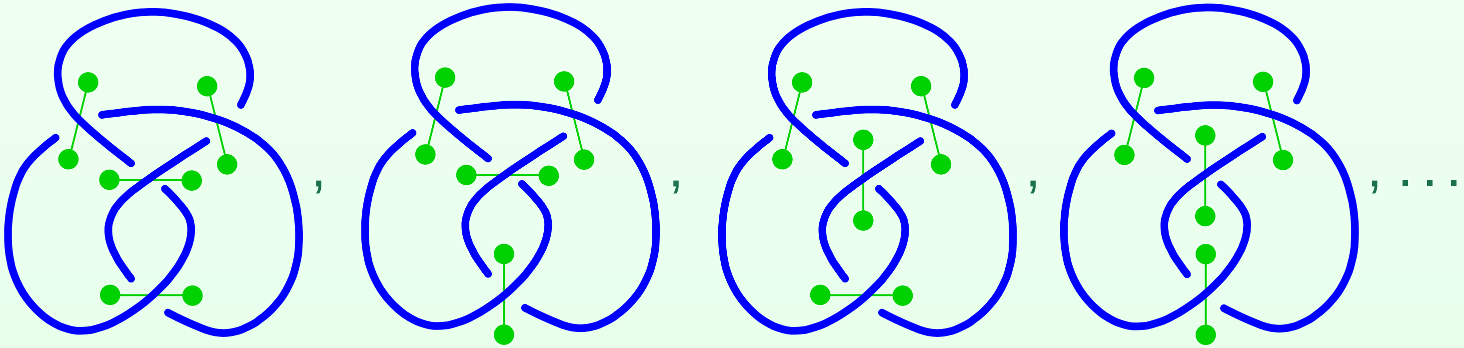
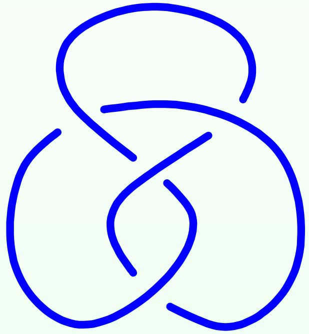
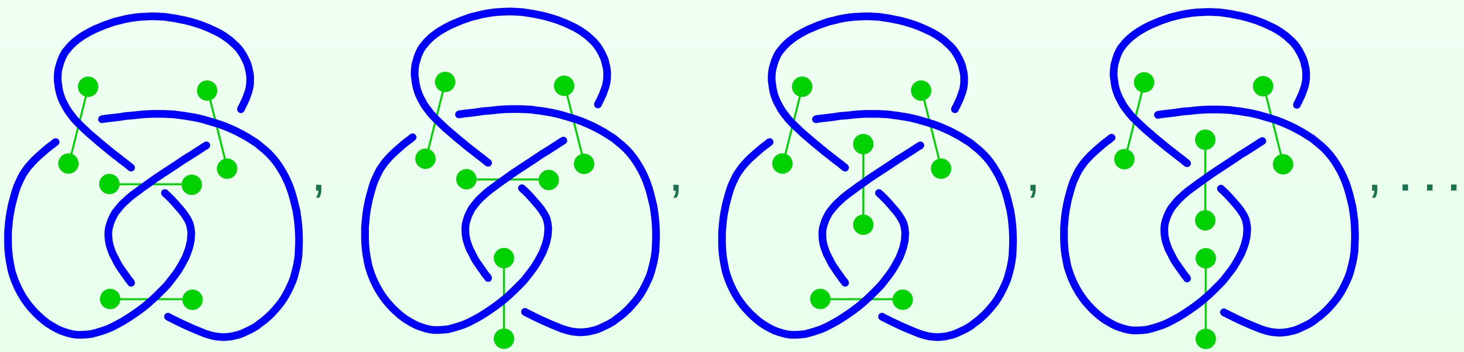
Knot diagram:  and its states:



Kauffman state sum. I

A *state* of diagram is a distribution of *markers* over all crossings.

Knot diagram:  and its states:



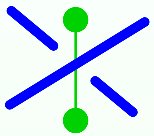
Totally 2^c states, where c is the number of crossings.

Kauffman state sum. II

Three numbers associated to a state s:

Kauffman state sum. II

Three numbers associated to a state s:

1. the number $a(s)$ of *positive* markers  ,

Kauffman state sum. II

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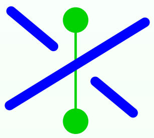
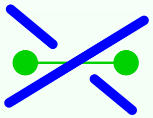
1. the number $a(s)$ of *positive* markers  ,
2. the number $b(s)$ of *negative* markers  ,

Kauffman state sum. II

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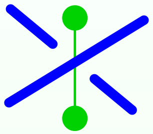
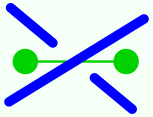
1. the number $a(s)$ of *positive* markers  ,
2. the number $b(s)$ of *negative* markers  ,
3. the number $|s|$ of components of the curve obtained by smoothing along the markers:

Kauffman state sum. II

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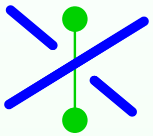
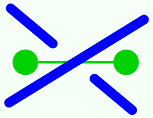
1. the number $a(s)$ of *positive* markers  ,

2. the number $b(s)$ of *negative* markers  ,

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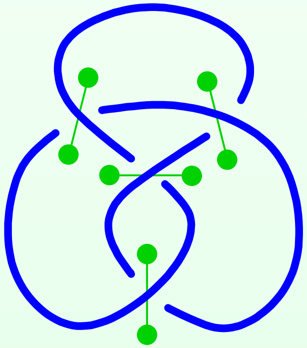
$s =$  $\mapsto$

Kauffman state sum. II

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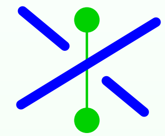
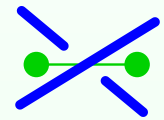
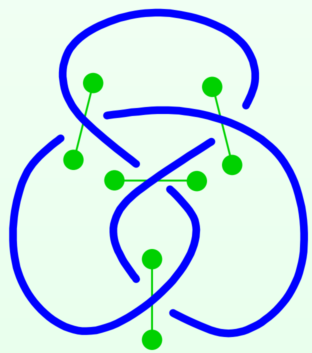
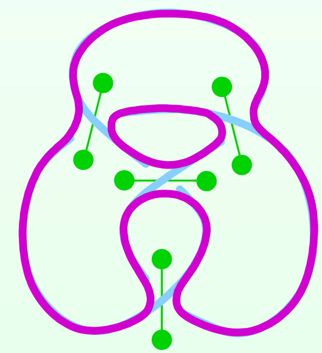
1. the number $a(s)$ of *positive* markers  ,

2. the number $b(s)$ of *negative* markers  ,

3. the number $|s|$ of components of the curve obtained by smoothing along the markers:

$s =$  $\mapsto$ smoothing$(s) =$ 

Kauffman state sum. II

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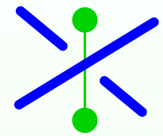
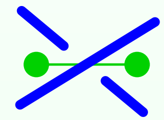
1. the number $a(s)$ of *positive* markers  ,
2. the number $b(s)$ of *negative* markers  ,
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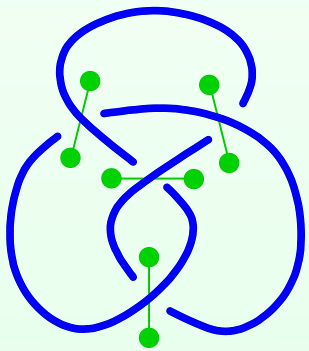
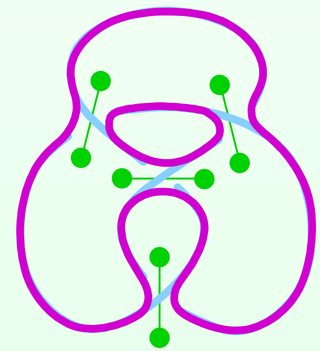
$s =$  $\mapsto$ smoothing$(s) =$ 

$|s| = 2$

Kauffman state sum. II

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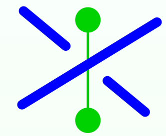
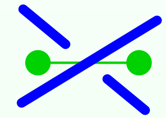
1. the number $a(s)$ of *positive* markers  ,
2. the number $b(s)$ of *negative* markers  ,
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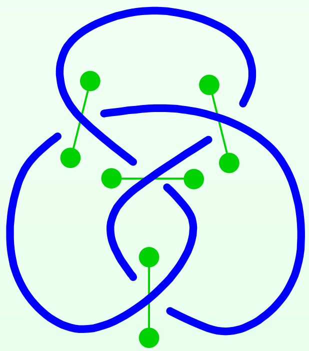
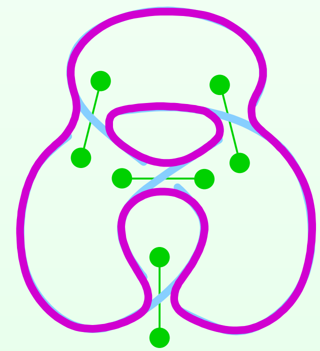
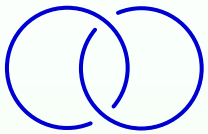
$s =$  $\mapsto$ smoothing$(s) =$ 

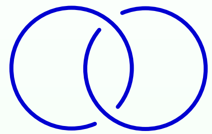
$|s| = 2$

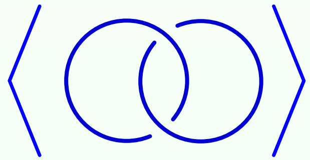
State Sum: $\langle D \rangle = \sum_{s \text{ state of } D} A^{a(s)-b(s)}(-A^2 - A^{-2})^{|s|}$

Example

Hopf link, 

Example

Hopf link, 

 $=$

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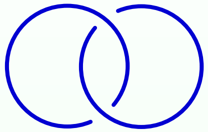
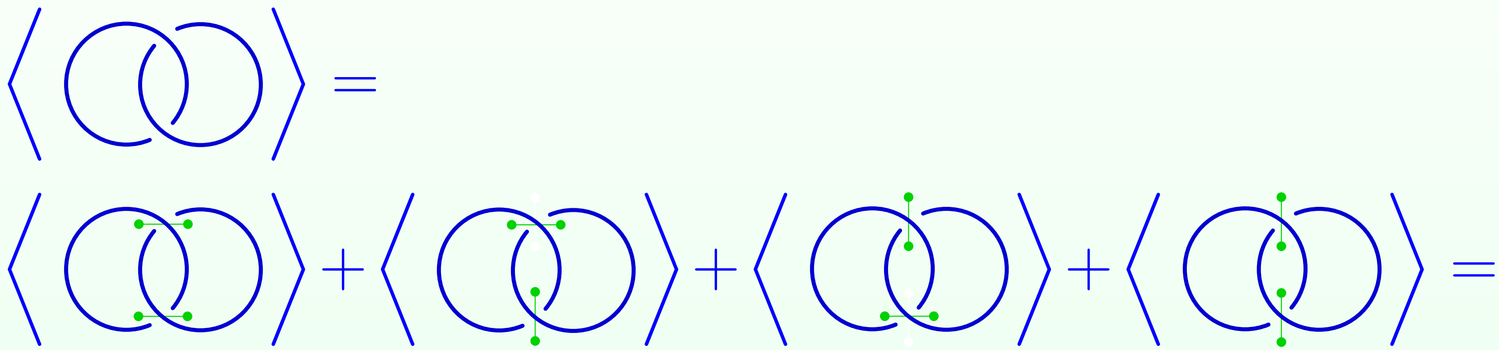
Hopf link, 



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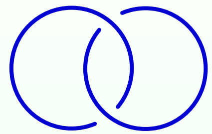
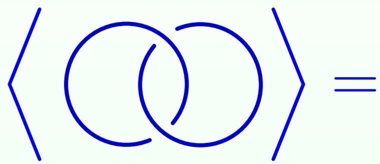
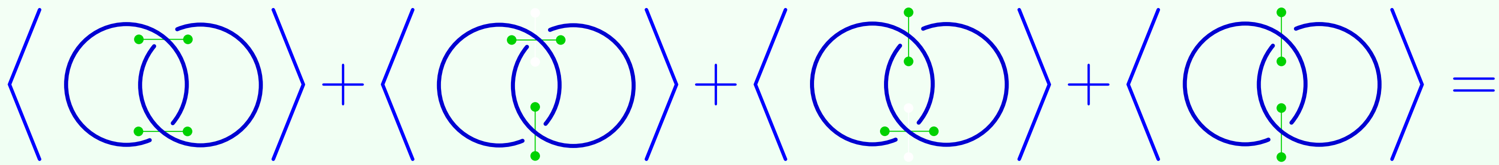
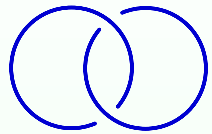
Hopf link, 

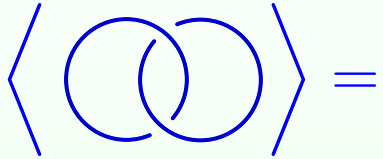


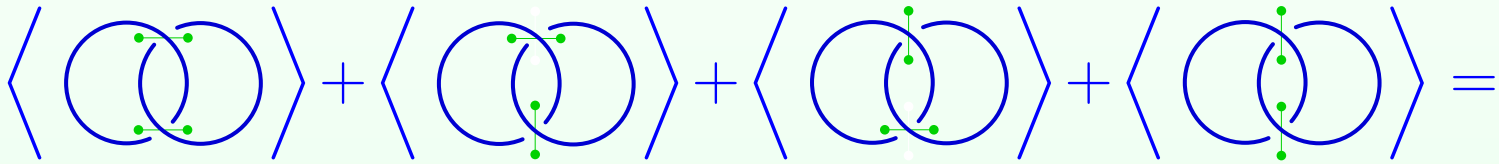


$$A^2(-A^2 - A^{-2})^2 + 2(-A^2 - A^{-2}) + A^{-2}(-A^2 - A^{-2})^2 =$$

Example

Hopf link, 

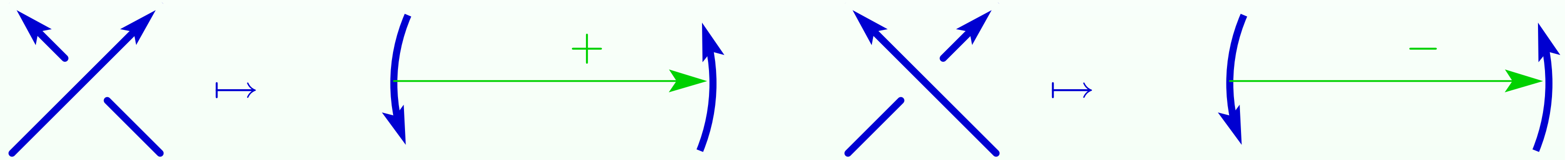




$$A^2(-A^2-A^{-2})^2+2(-A^2-A^{-2})+A^{-2}(-A^2-A^{-2})^2=$$
$$A^6+A^2+A^{-2}+A^{-6}$$

Kauffman state sum model for Gauss diagrams

Crossing $\longmapsto$ arrow.



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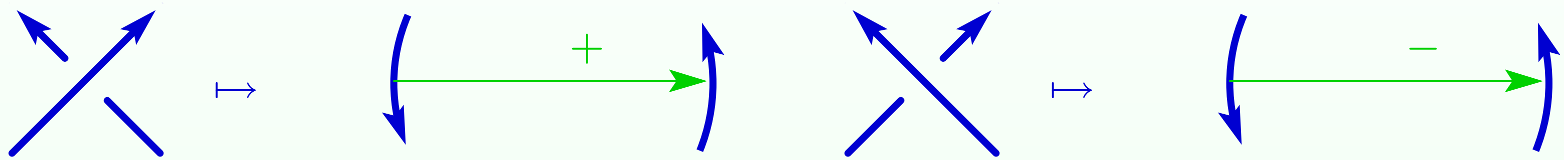

Smoothing of a crossing $\mapsto$ a surgery along the arrow.

Kauffman state sum model for Gauss diagrams

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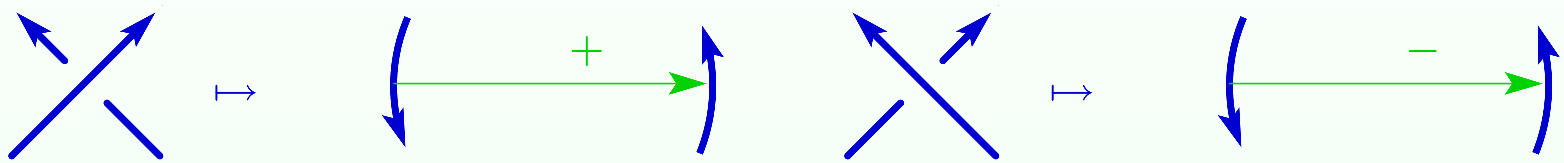

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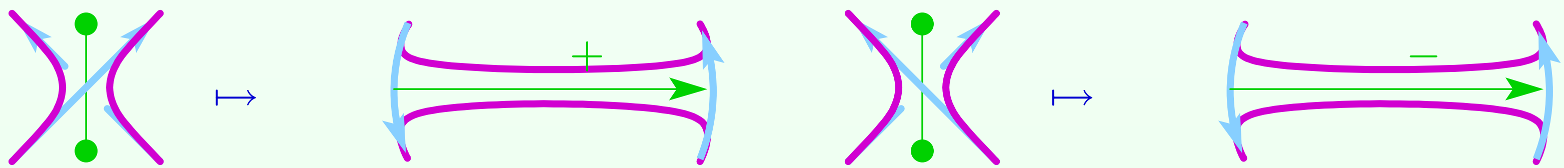

positive marker, positive crossing

negative marker, negative crossing

Kauffman state sum model for Gauss diagrams

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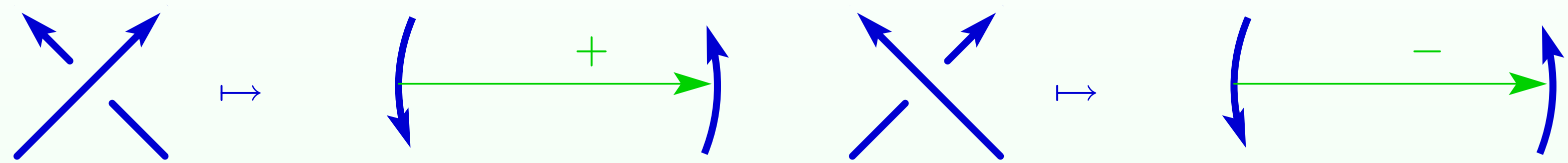

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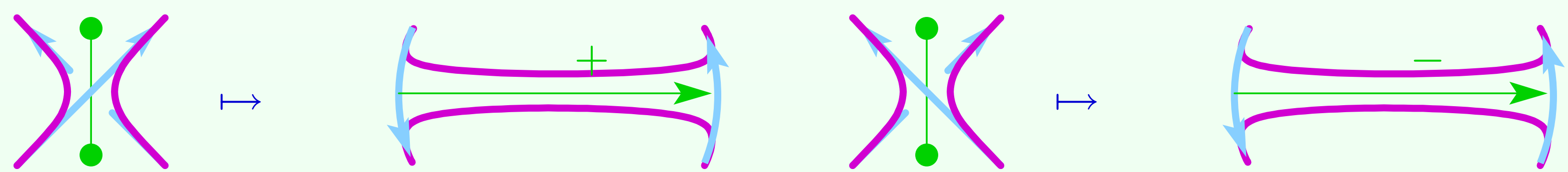

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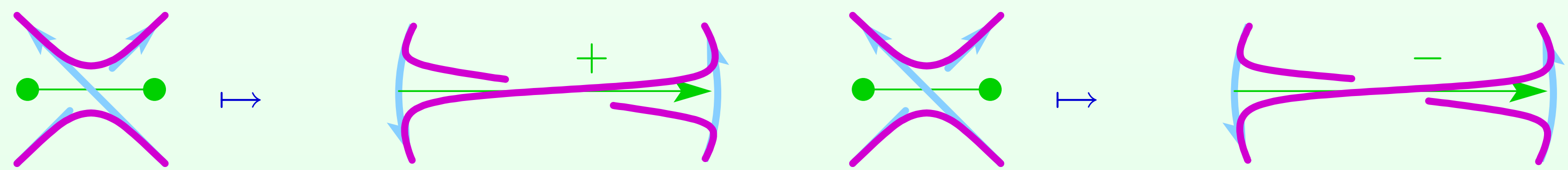

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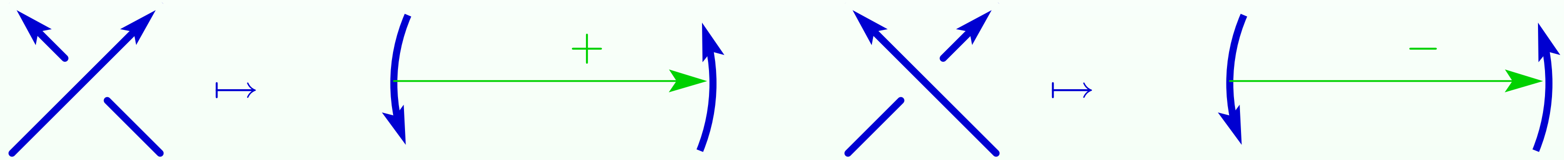

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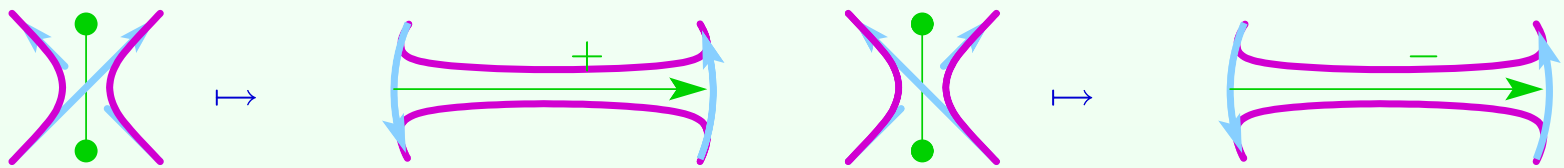
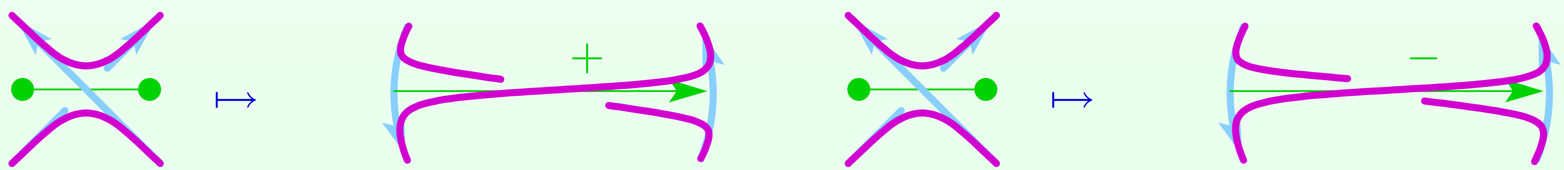


positive marker, positive crossing   negative marker, negative crossing



negative marker, positive crossing   positive marker, negative crossing

Smoothing depends only of the signs of marker and crossing.

Kauffman state sum model for Gauss diagrams

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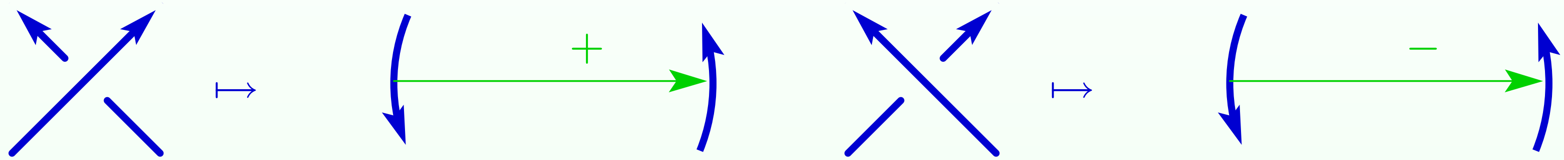
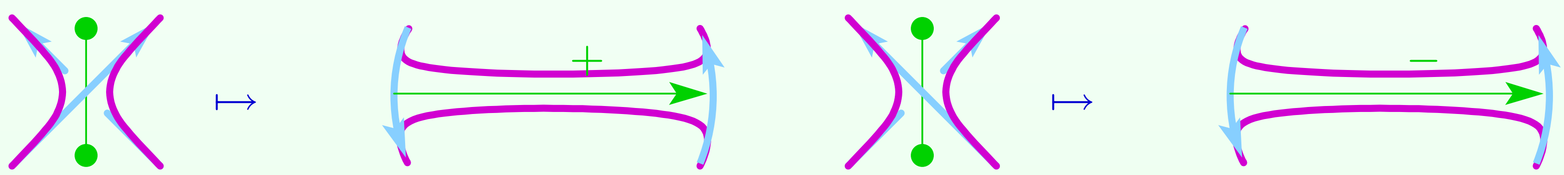
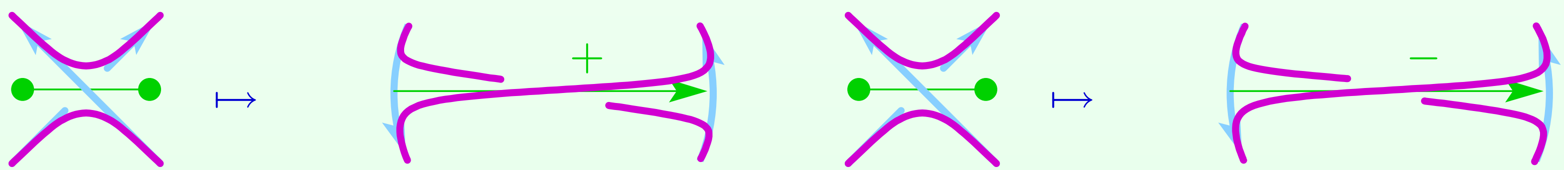


Smoothing of a crossing $\mapsto$ a surgery along the arrow.



positive marker, positive crossing — negative marker, negative crossing



negative marker, positive crossing — positive marker, negative crossing

Smoothing depends only of the signs of marker and crossing.

No need in direction of the arrow!

Kauffman state sum model for Gauss diagrams

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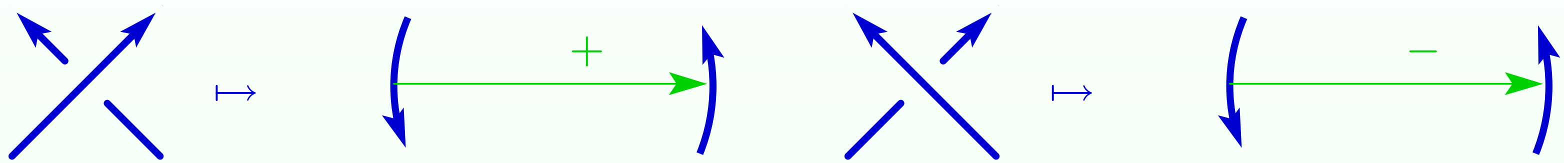

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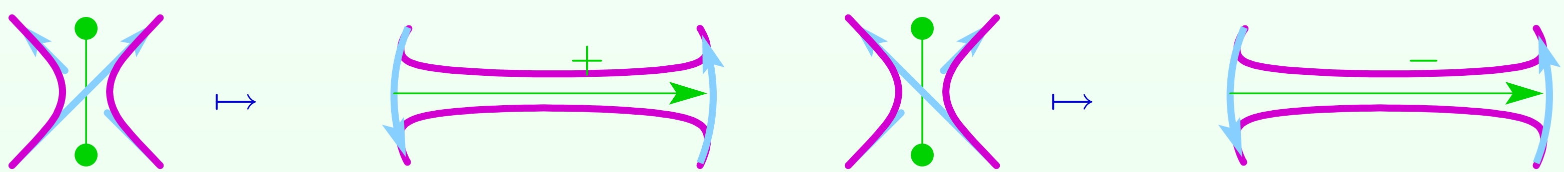

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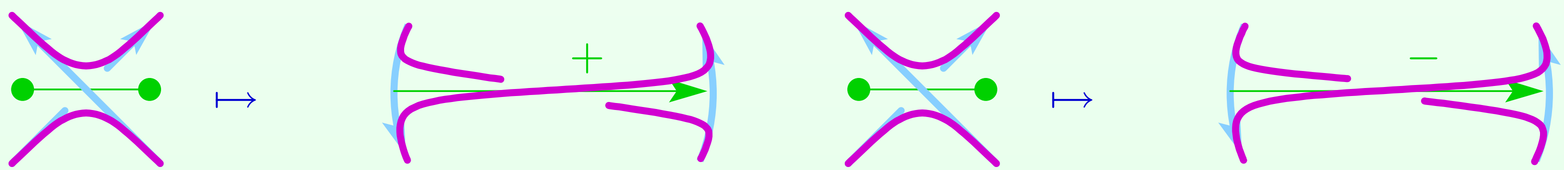
negative marker, negative crossing



negative marker, positive crossing

positive marker, negative crossing

Smoothing depends only of the signs of marker and crossing.

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Kauffman state sum is defined for *signed chord diagrams.*

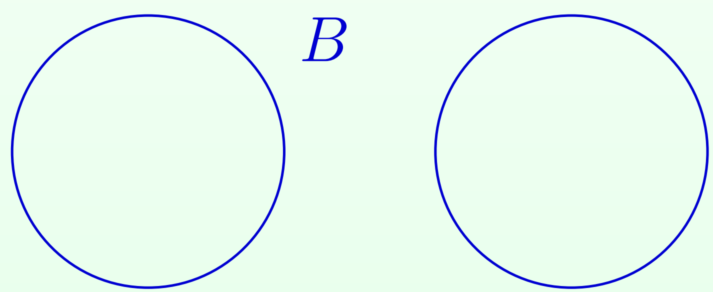
Gauss diagrams of a poor man

Signed chord diagrams

A chord diagram $(B, c_1, \dots, c_n)$

Signed chord diagrams

A chord diagram $(B, c_1, \dots, c_n)$
(a closed 1-manifold B (*base*), and



Signed chord diagrams

A chord diagram $(B, c_1, \dots, c_n)$
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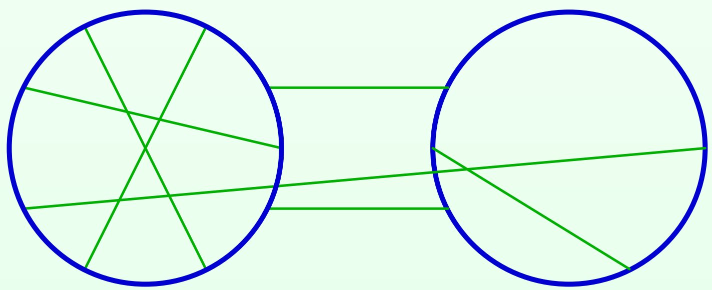
disjoint chords $c_1, \dots, c_n$ with end points on the base.)



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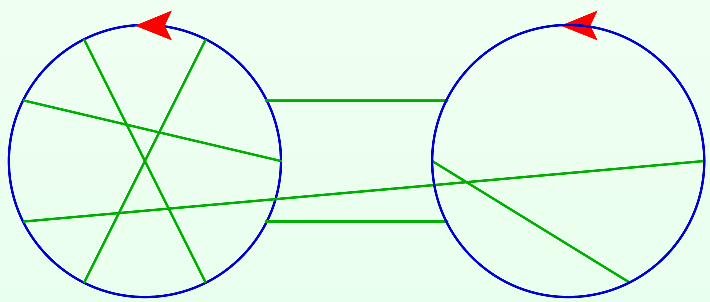
● in which B is oriented and



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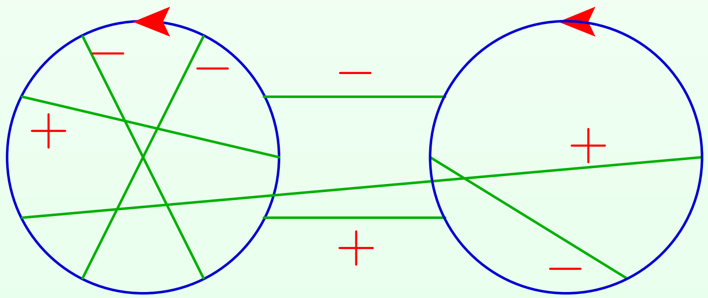
- each chord is equipped with a sign



Signed chord diagrams

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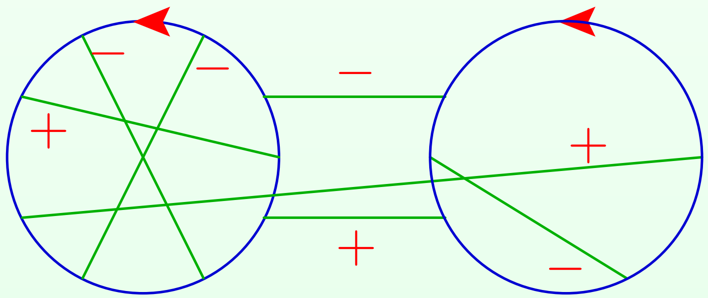
● in which B is oriented and
● each chord is equipped with a sign
is called a *signed chord diagram.*



State of signed chord diagram

A *state* of the signed chord diagram
is a distribution of another collection of signs over the set of all chords.

State of signed chord diagram

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These are *marker signs*,

State of signed chord diagram

A *state* of the signed chord diagram
is a distribution of another collection of signs over the set of all chords.
These are *marker signs*, the original signs are *structure signs*.

Smoothing of a signed chord diagram

A *smoothing* of a chord diagram $(B, c_1, \dots, c_n)$ is the result of Morse modifications of index 1 performed on B along each of its chords.

Smoothing of a signed chord diagram

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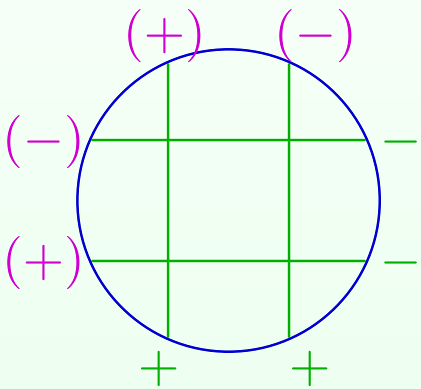

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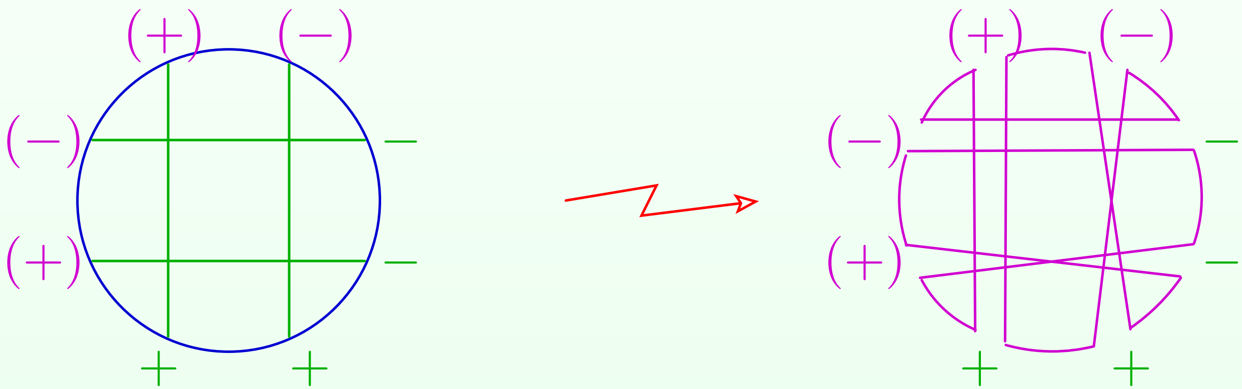

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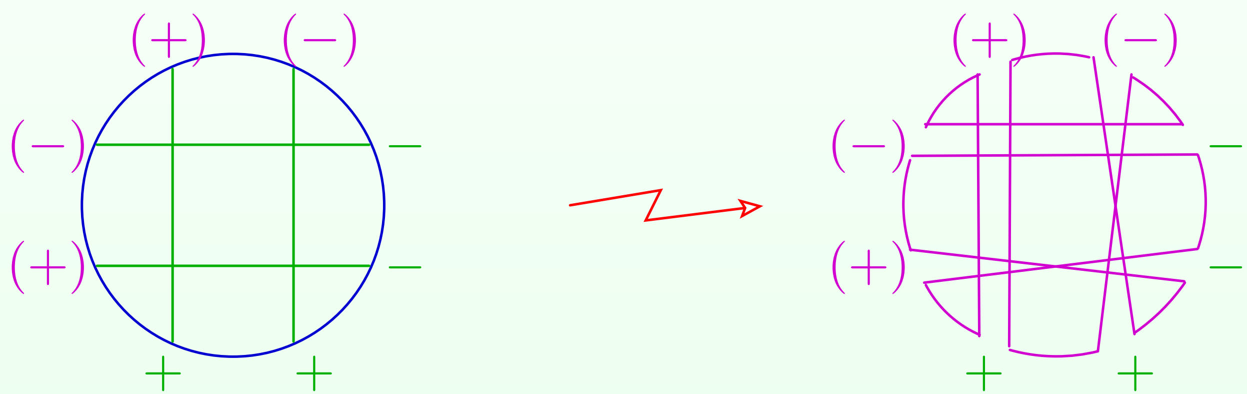

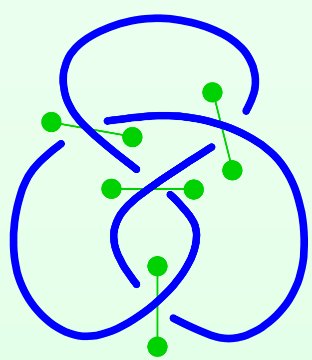

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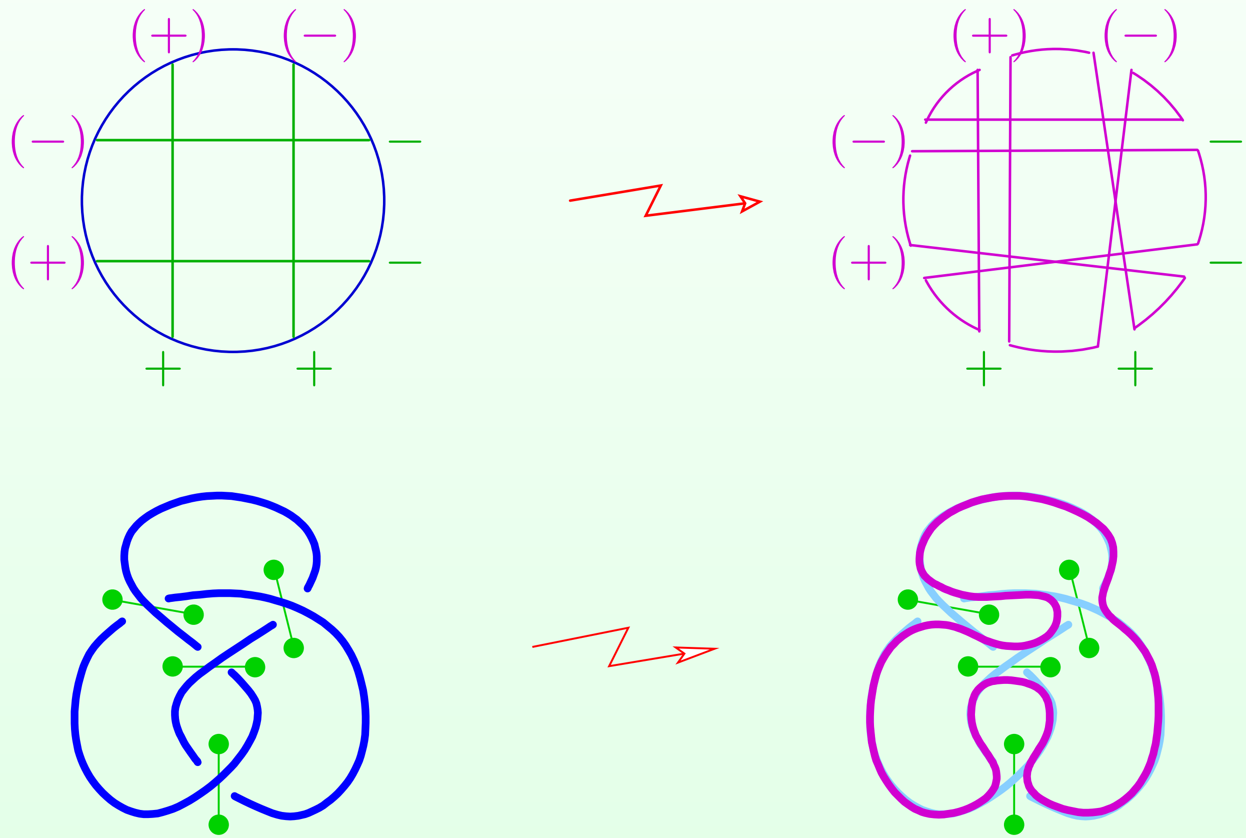

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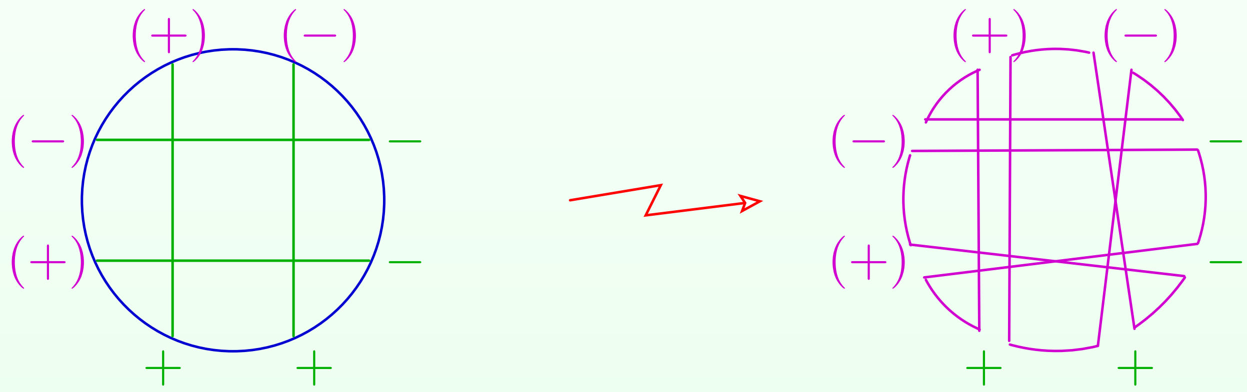

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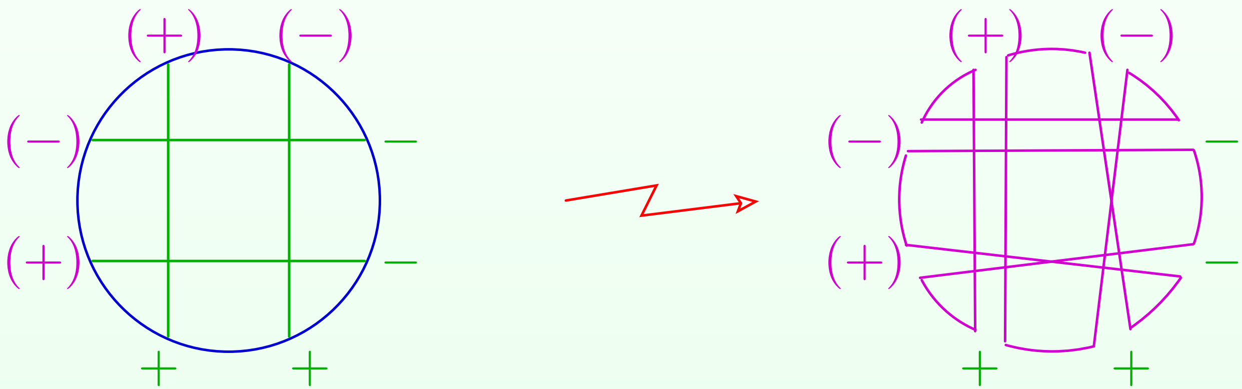

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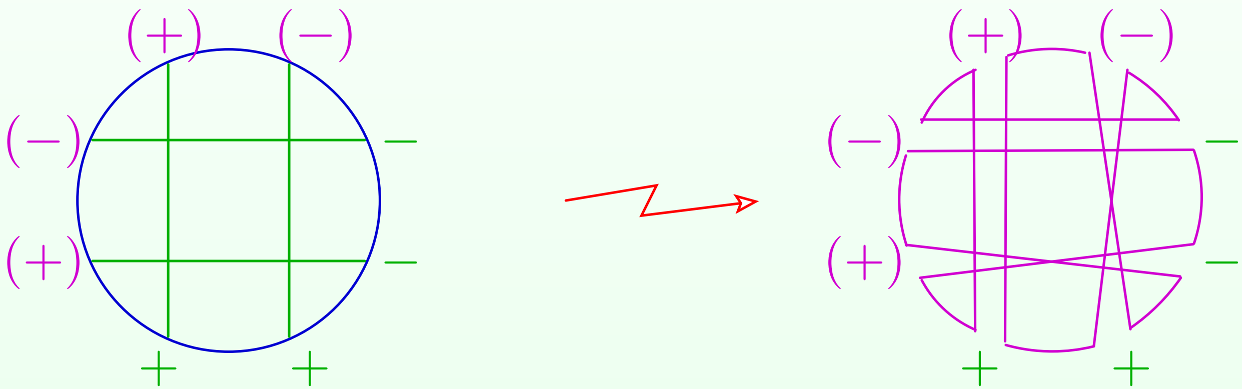

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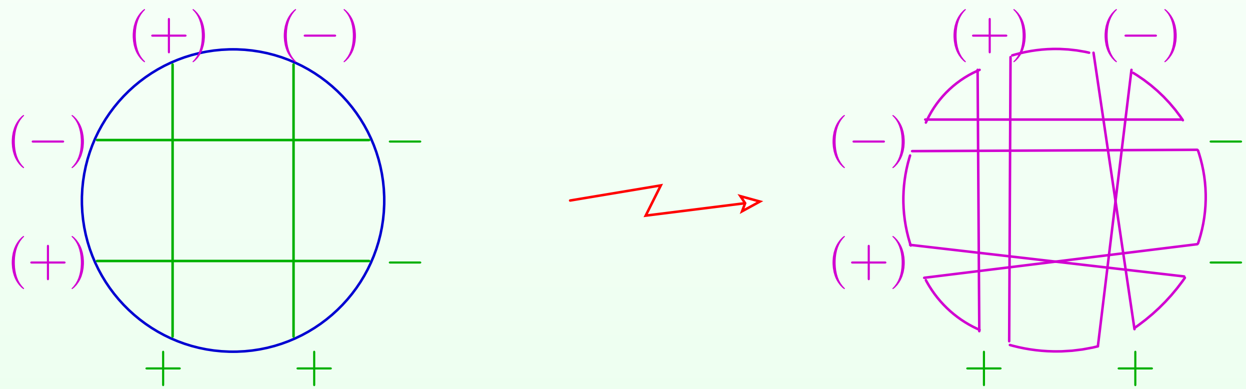

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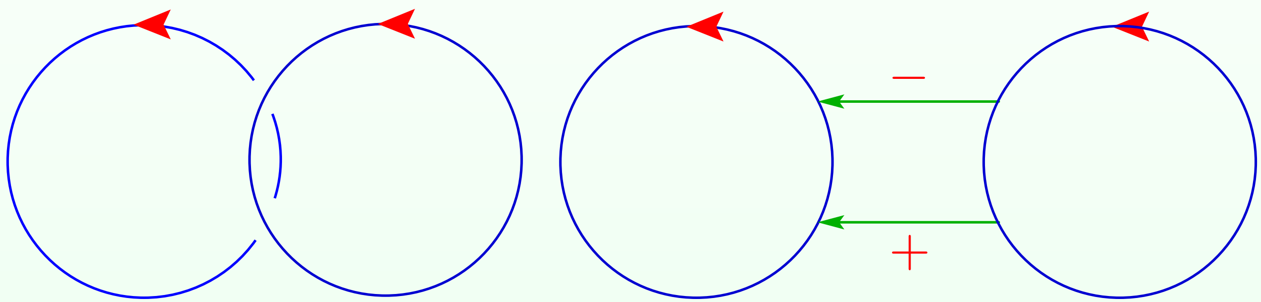

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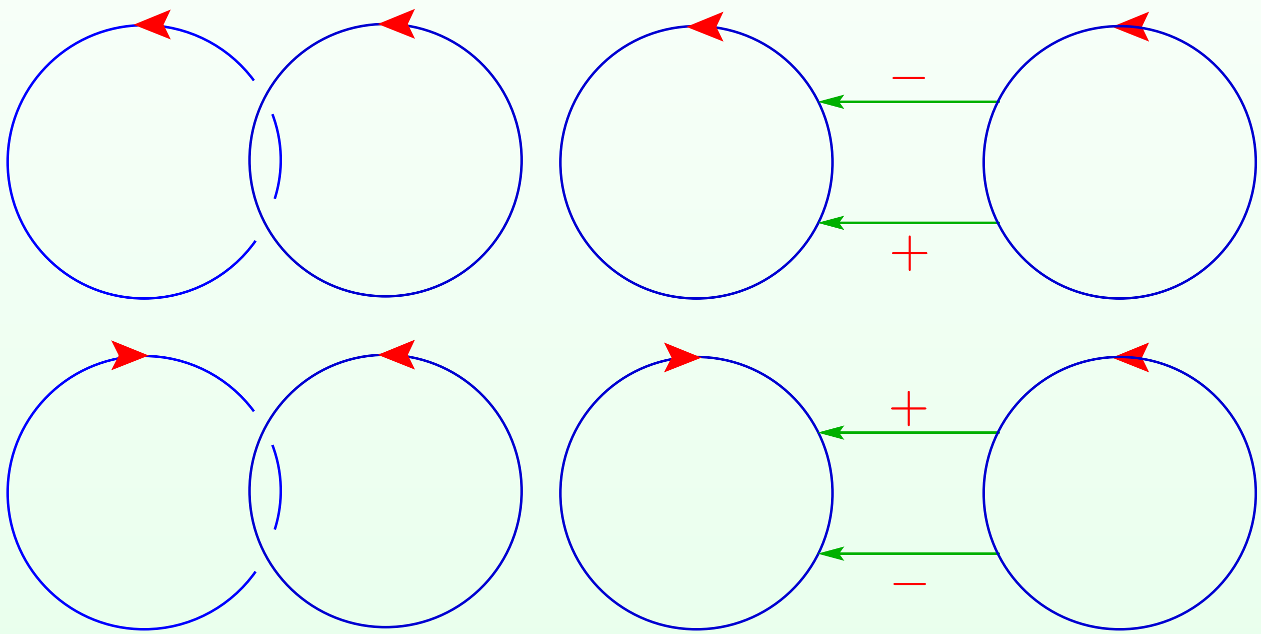

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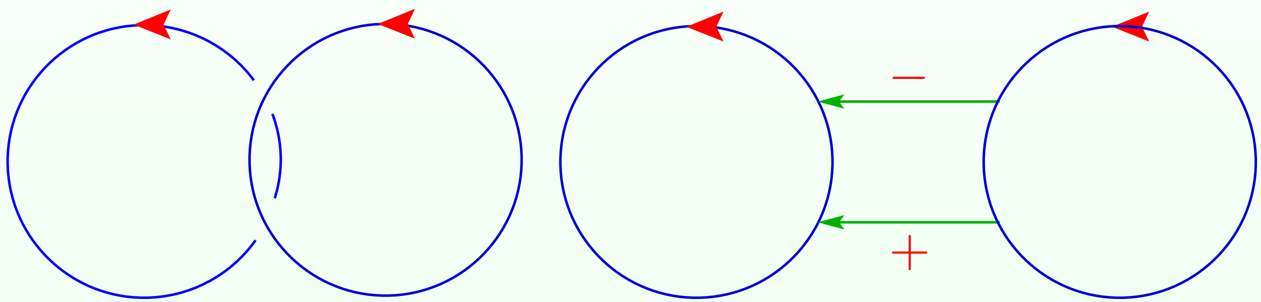

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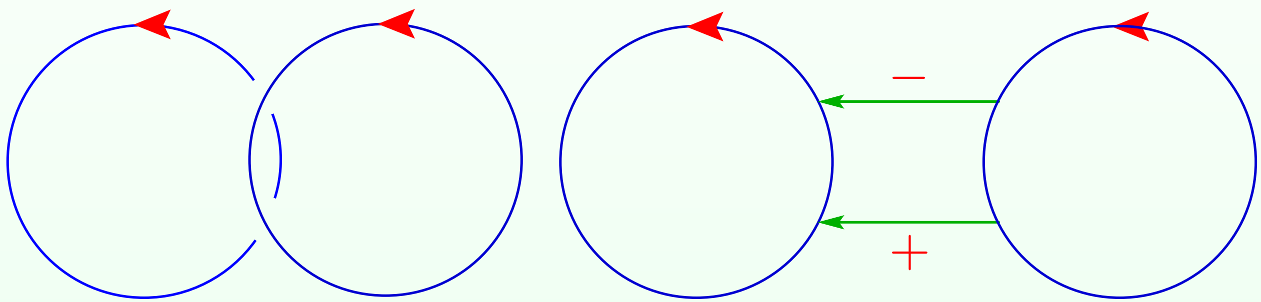

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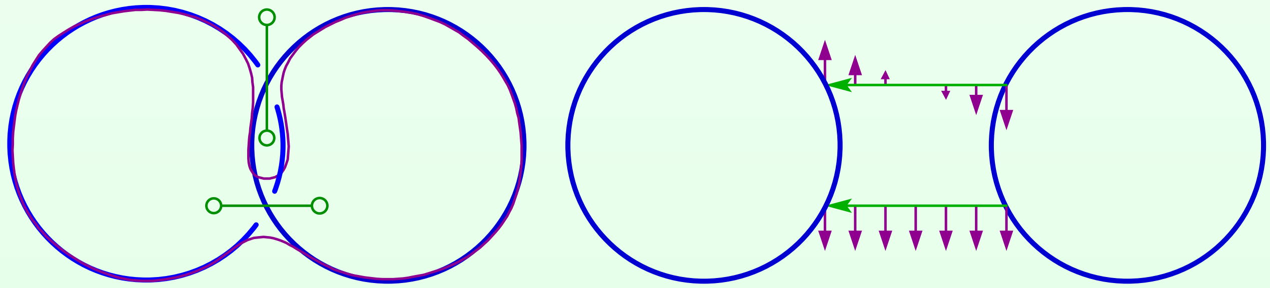

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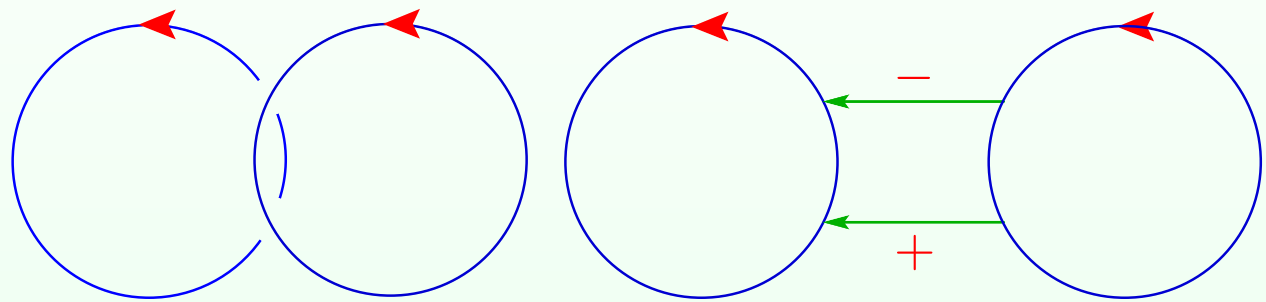

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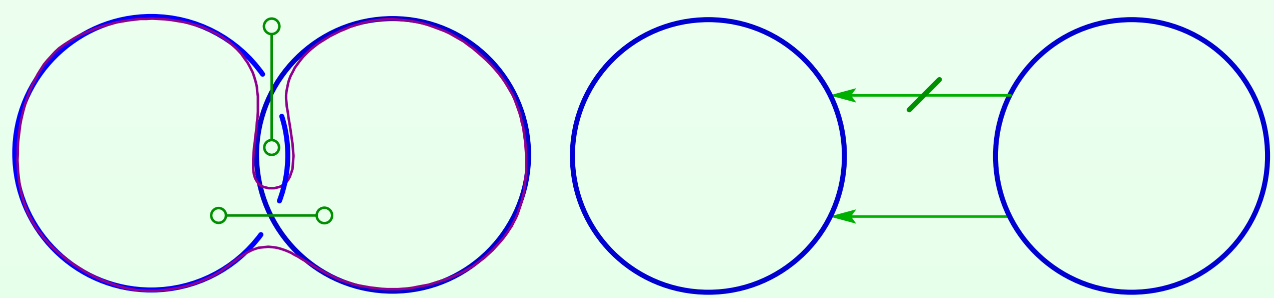

shorthand notation

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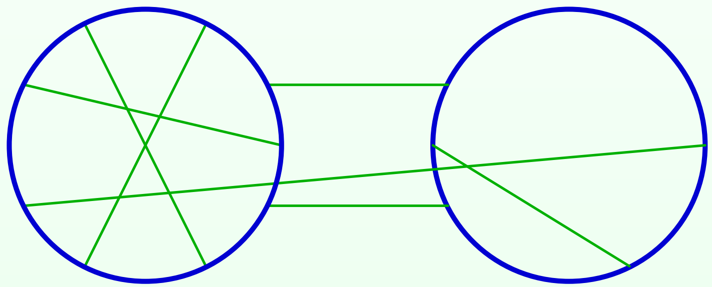

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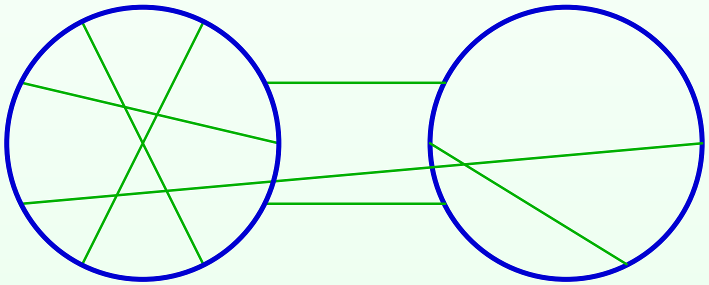

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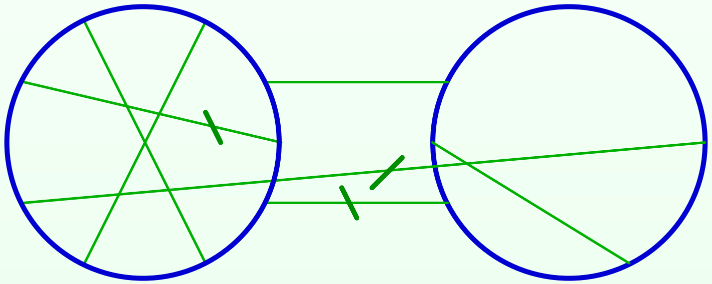

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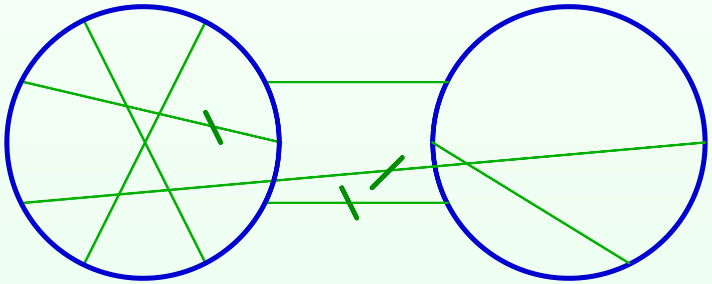

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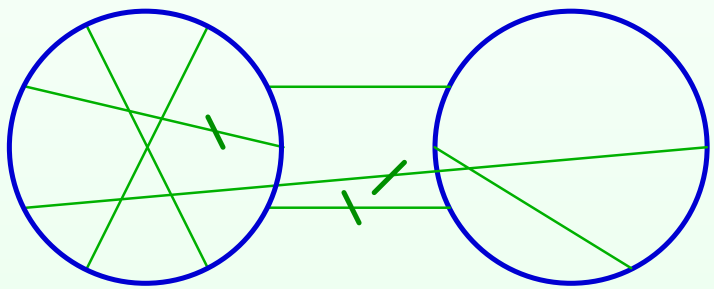

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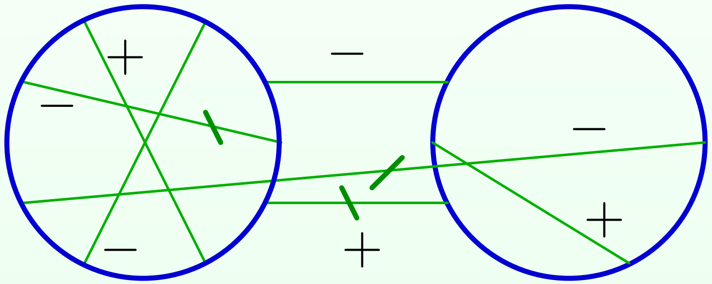

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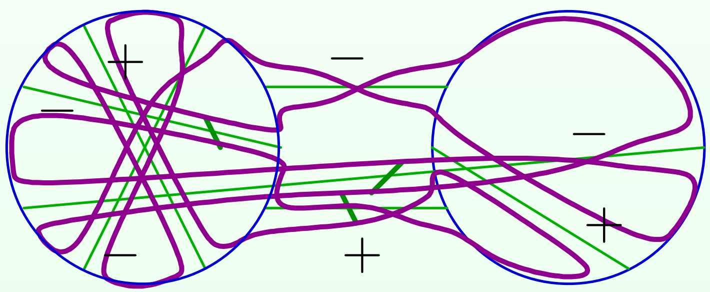

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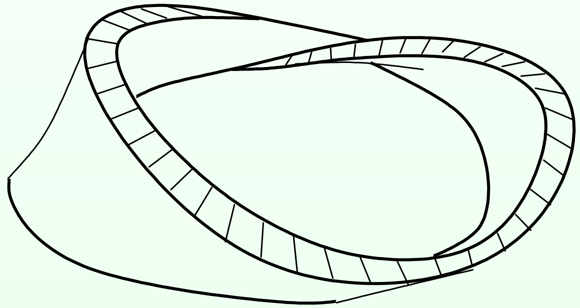

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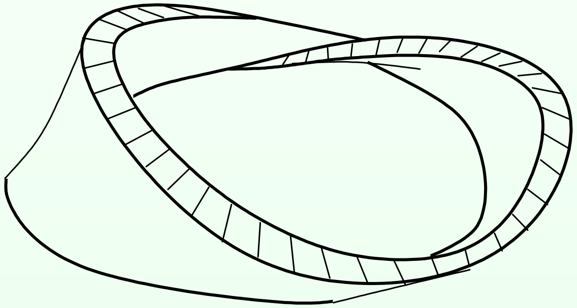

A neighborhood of M in $\mathbb{R}^3$ is orientable and fibers over M.

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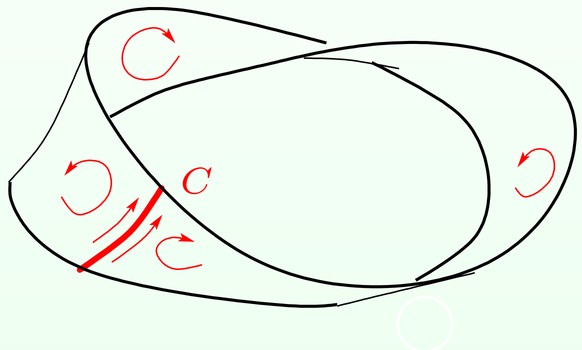

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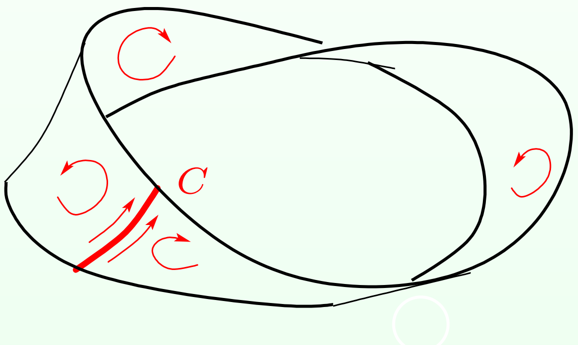

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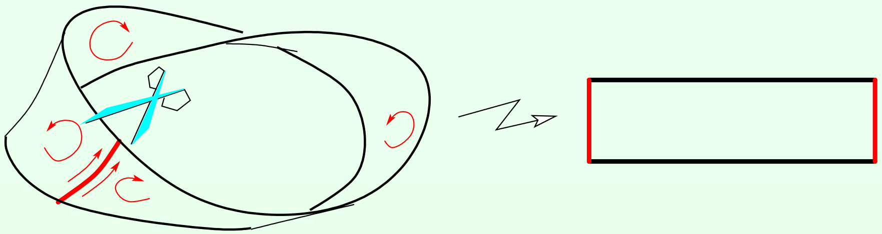

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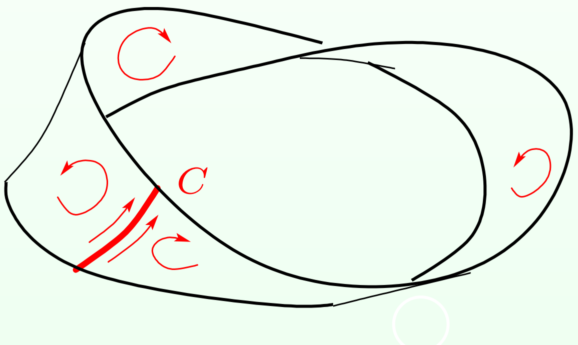

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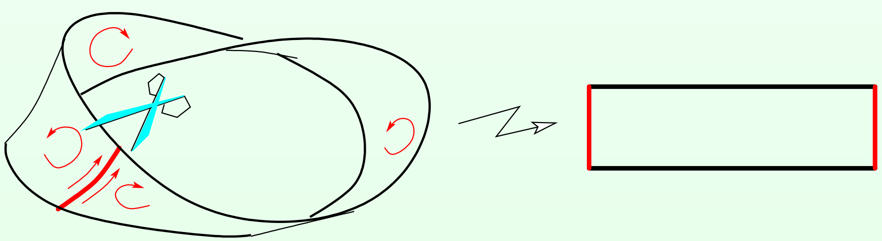

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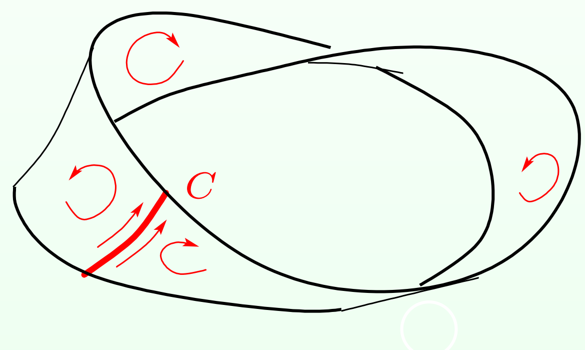

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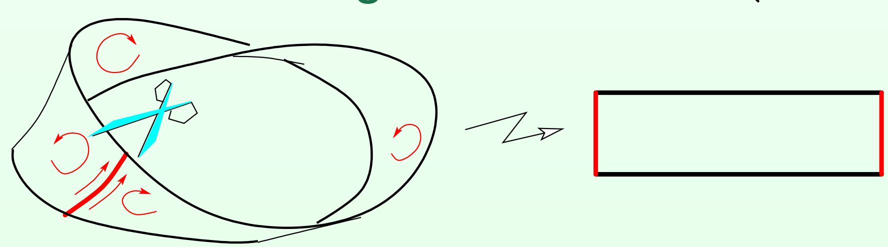

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4. Paste over the sides of the cut $(x_+, t) \sim (x_-, -t)$.

A link in orientable thickening of a non-orientable surface

A diagram on the surface.

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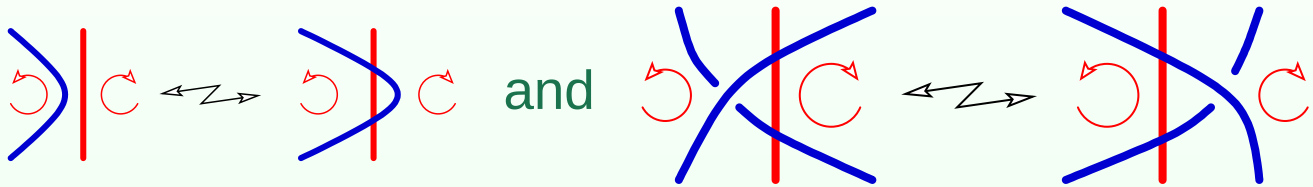
Reidemeister moves plus two more moves:



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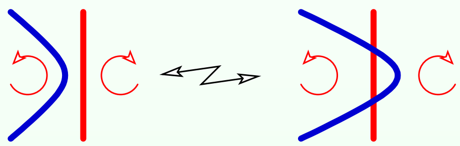

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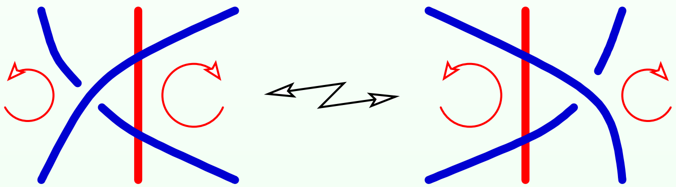

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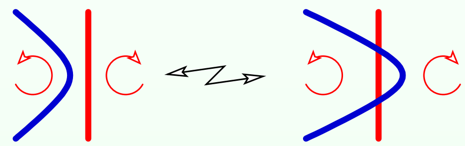
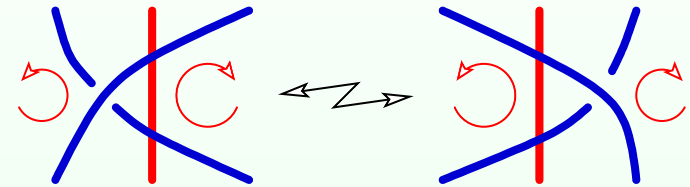
 and 

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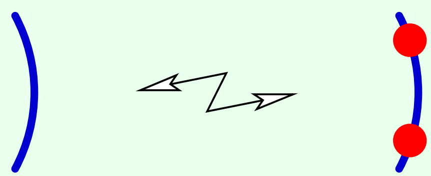
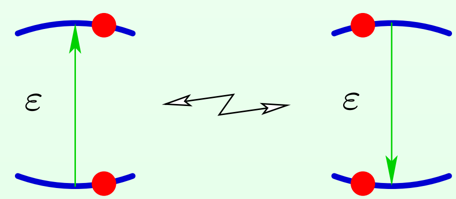
 and 

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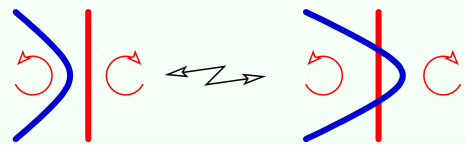

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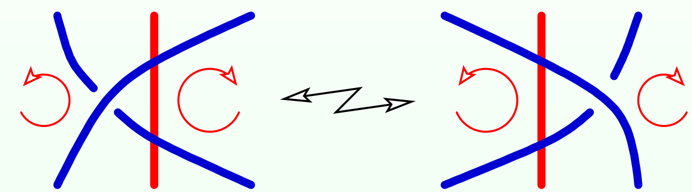

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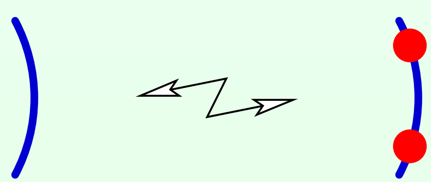

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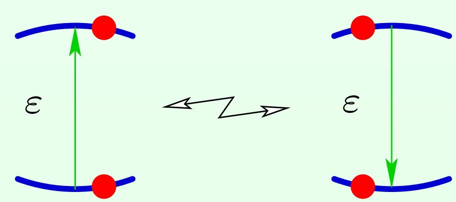

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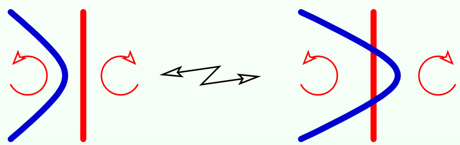
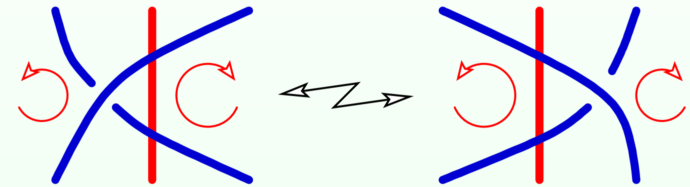
 and 

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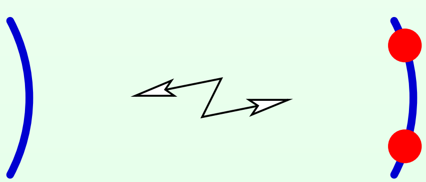
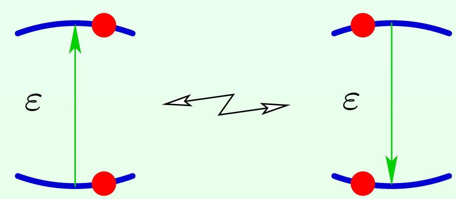
 and 

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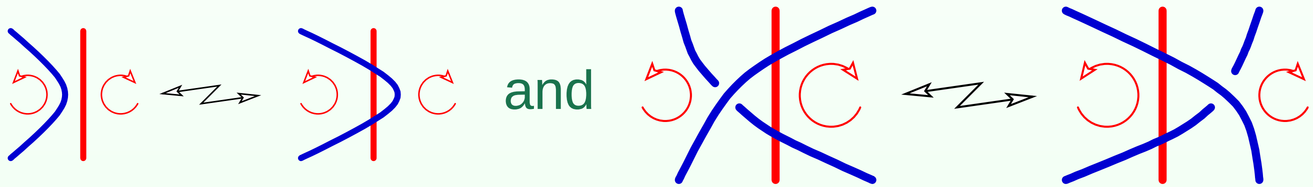

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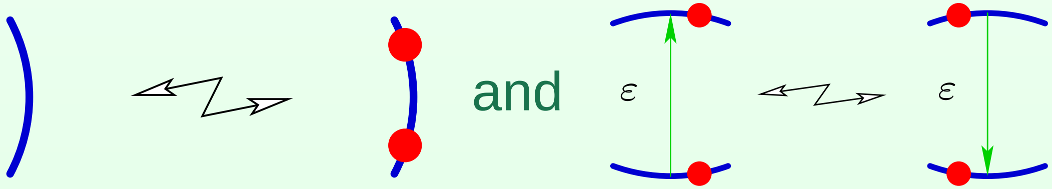

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Corollary (Bourgoin). *Links in orientable thickenings of surfaces have well-defined Kauffman bracket.*

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In other words: $H_{p,q}(D) = \mathcal{H}^{i,j}(D)$ iff
$q + 2j = 3w(D)$ and $j - i + p = w(D)$.

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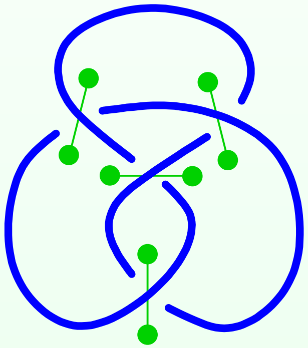
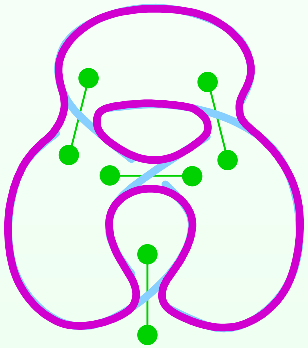
For example: state  with smoothing 

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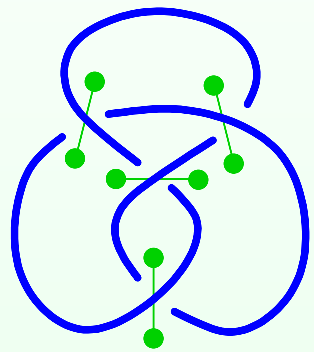
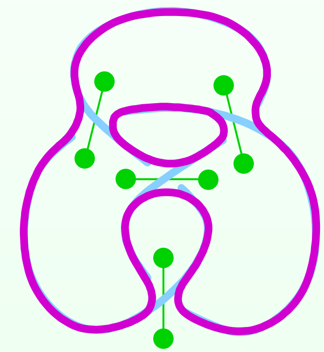
For example: state  with smoothing 

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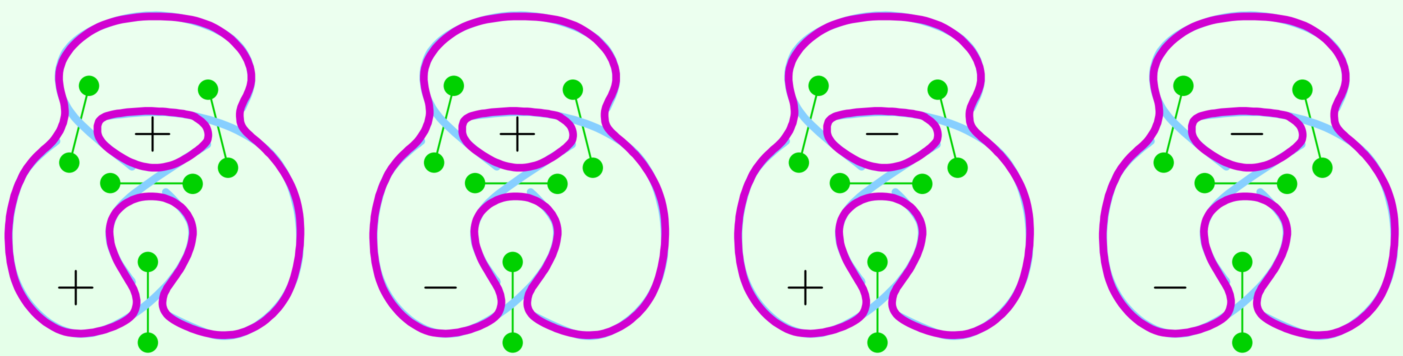

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Differentials are defined by the multiplication and co-multiplication in $\mathcal{A}$.

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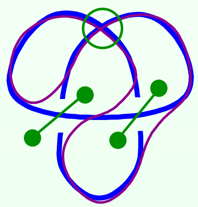
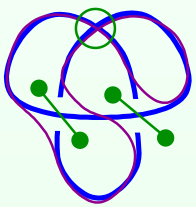
 $-A^4 - 1 \longrightarrow -A^2 - A^{-2}$ 

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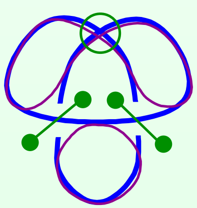
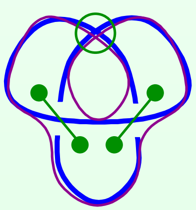
 $A^4 + 2 + A^{-4} \longrightarrow -1 - A^{-4}$ 

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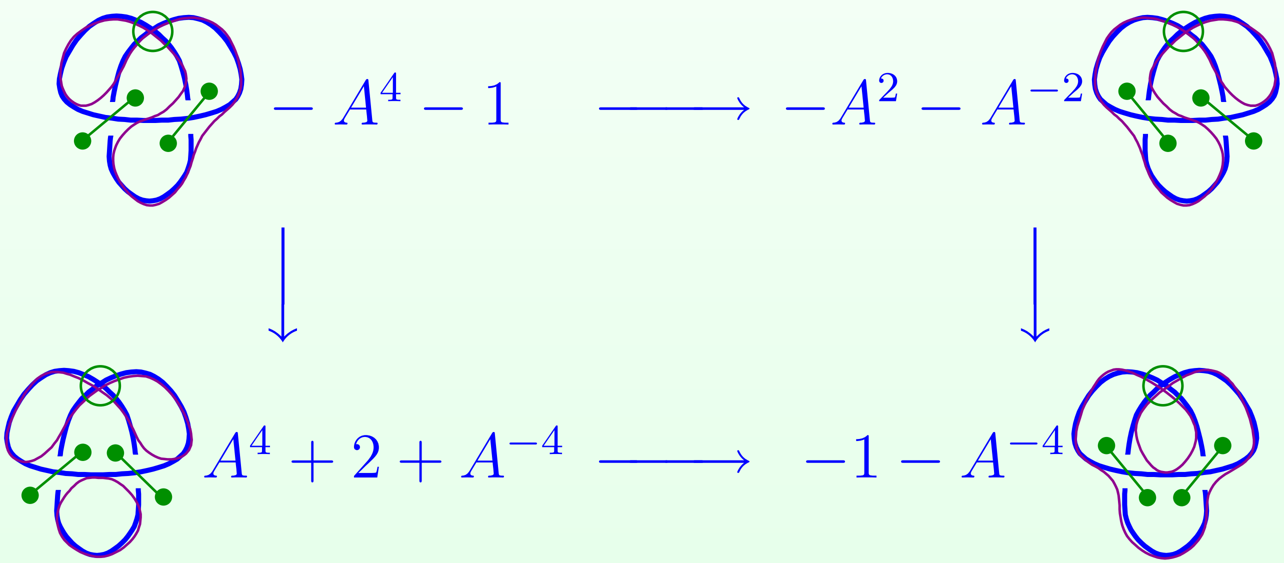

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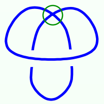
Consider virtual diagram of the unknot: 

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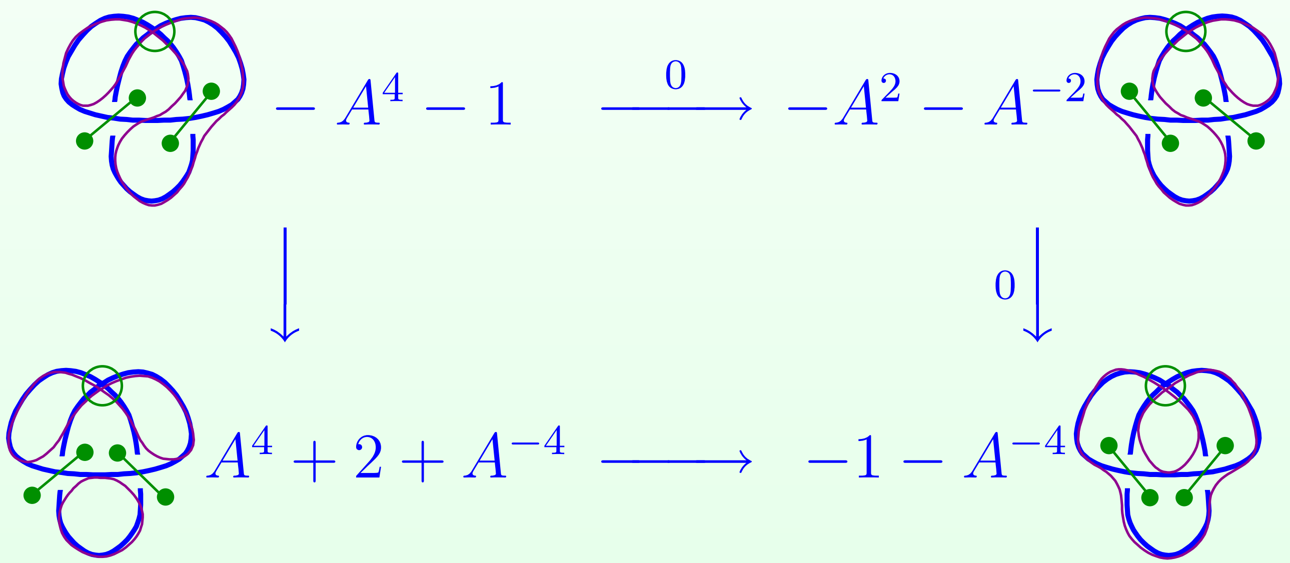

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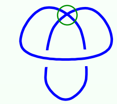
Consider virtual diagram of the unknot: 

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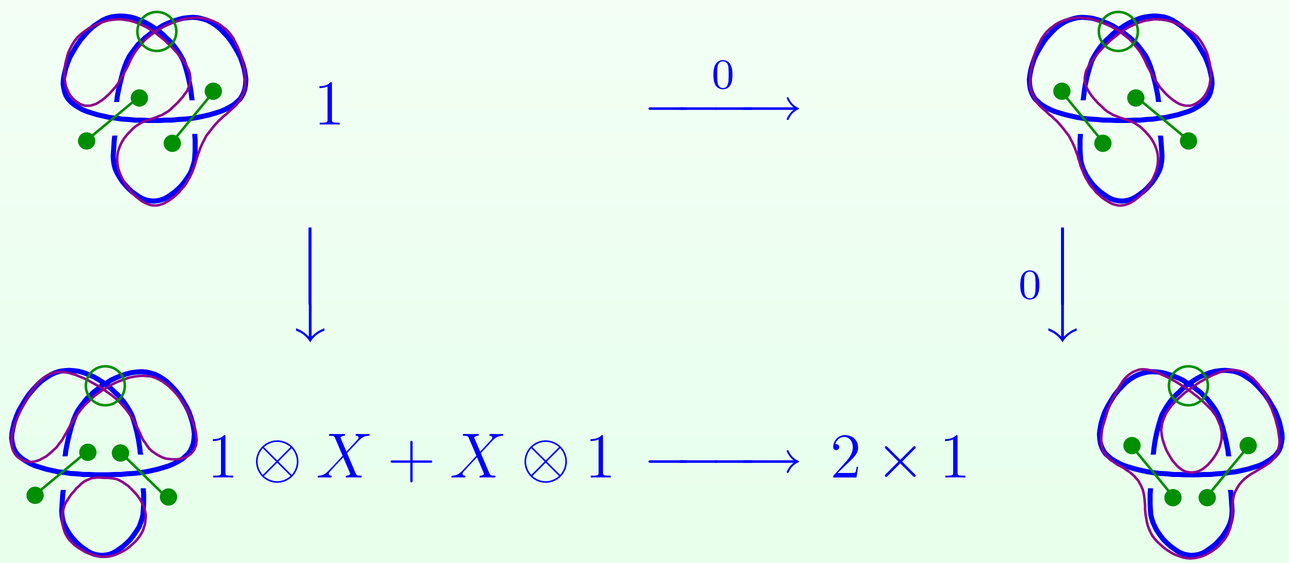

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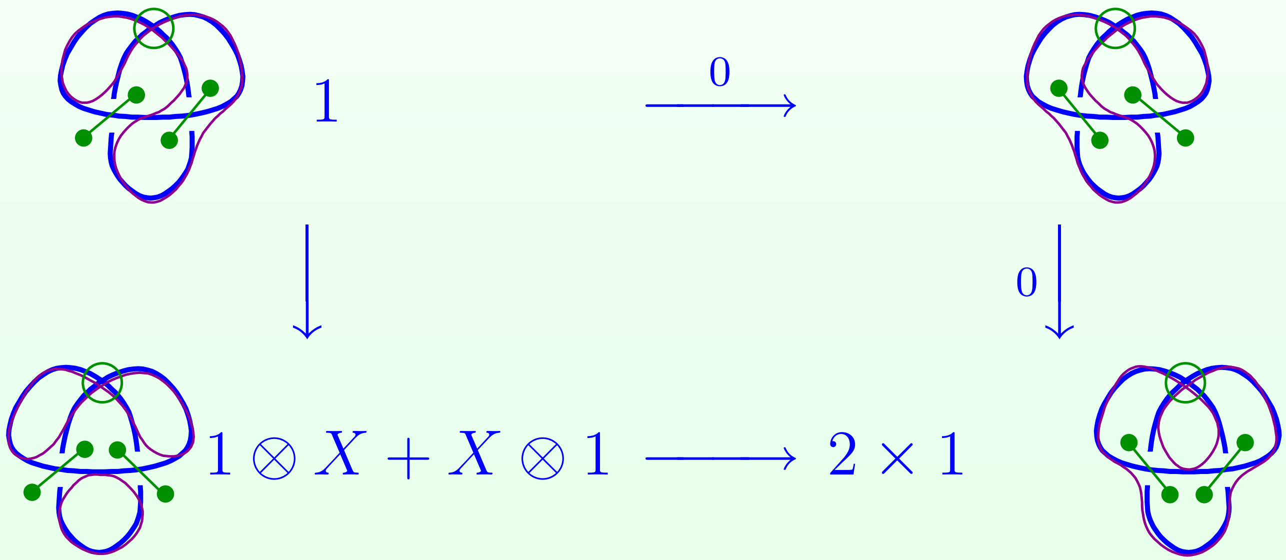

This does not happen if the chord diagram is *orientable*!

Orientation of chord diagrams

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= orientations of chords and arcs

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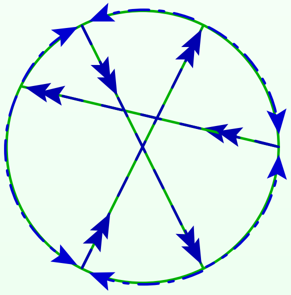

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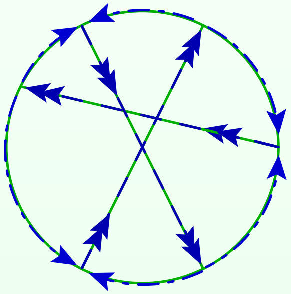

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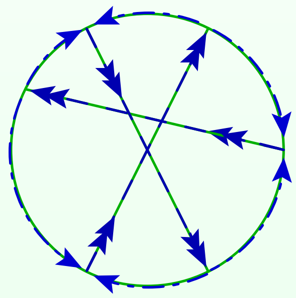

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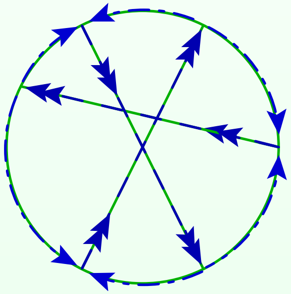

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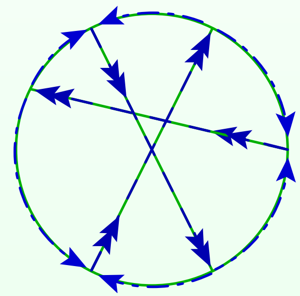

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The simplest nonorientable chord diagram: $\otimes$.

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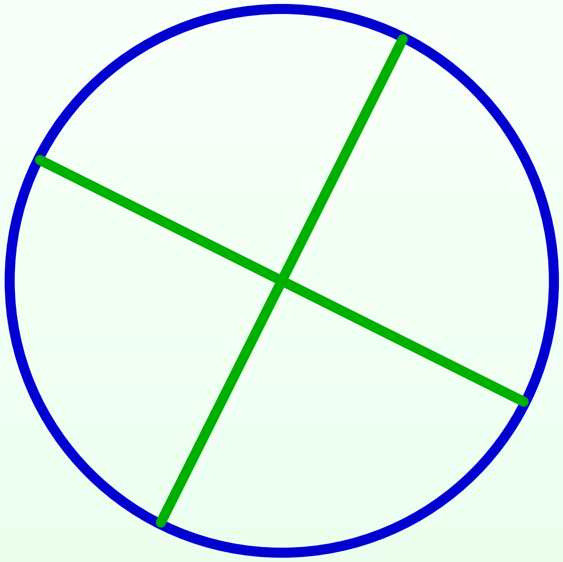

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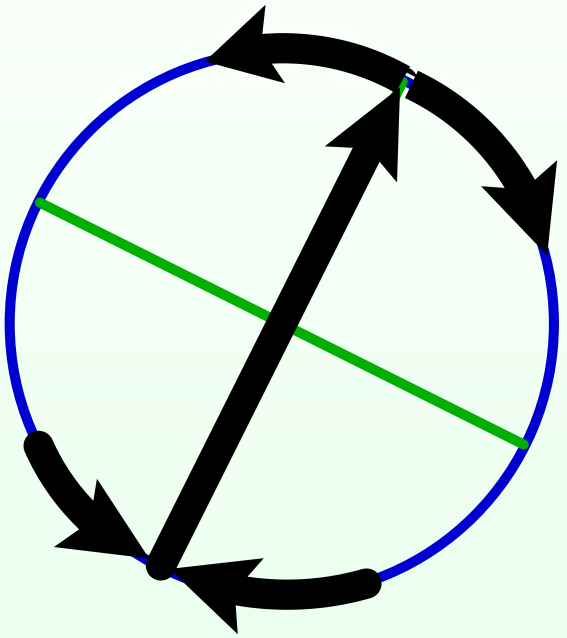

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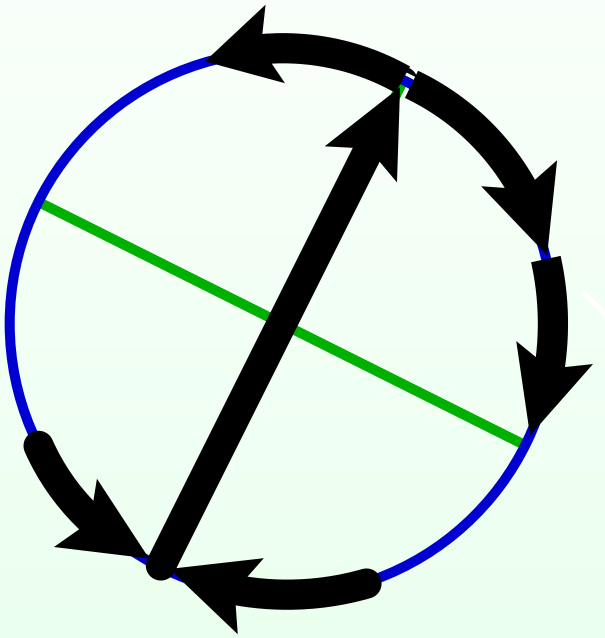

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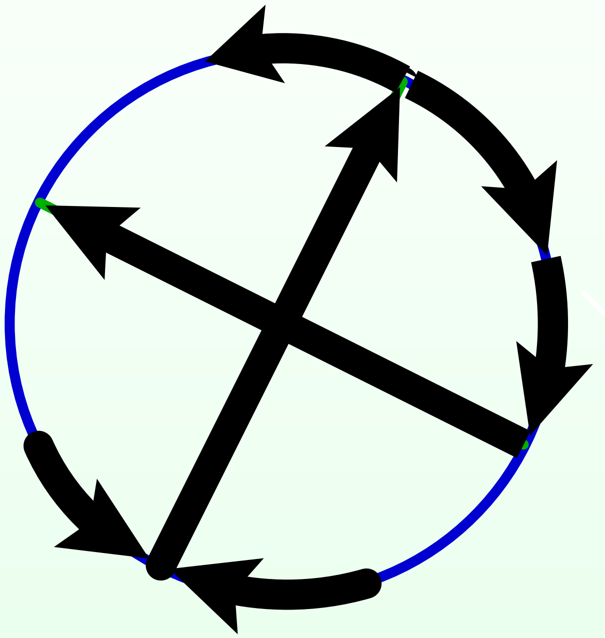

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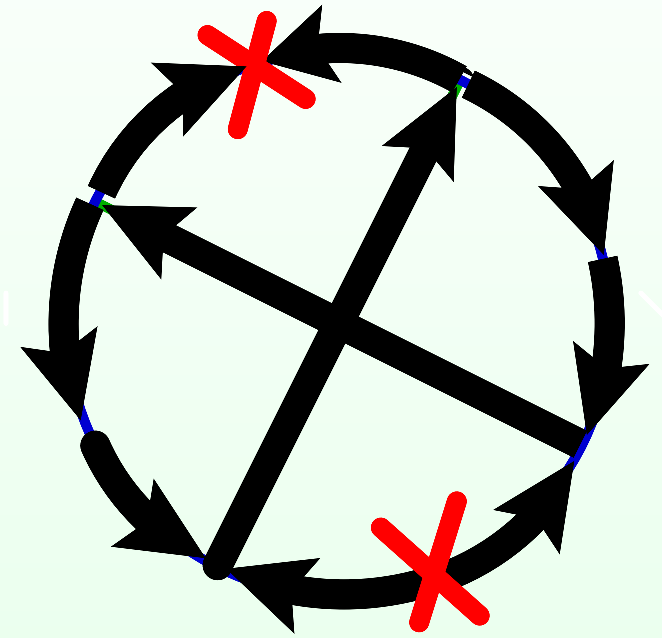

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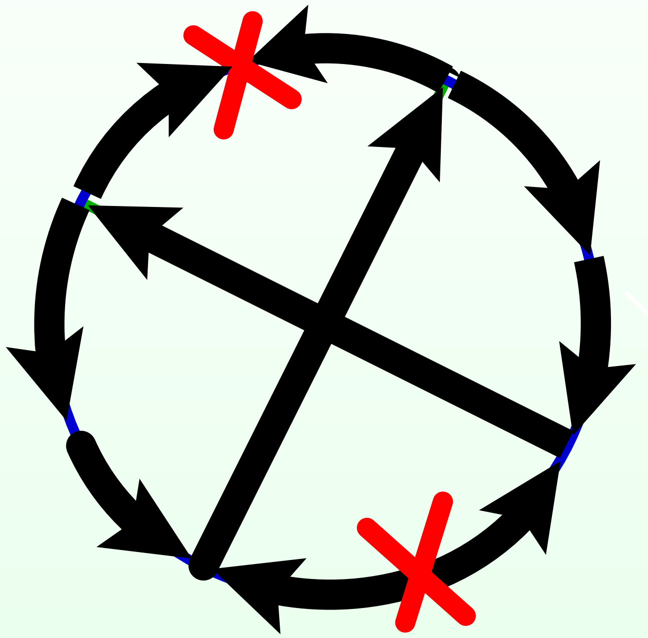

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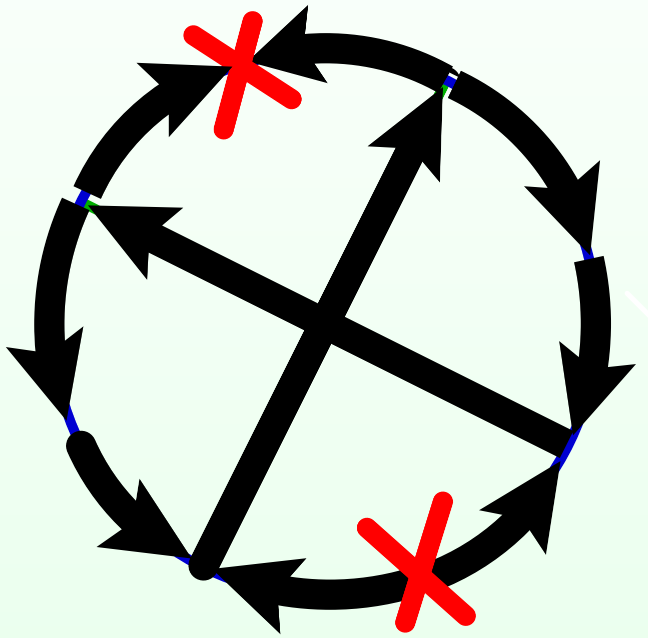

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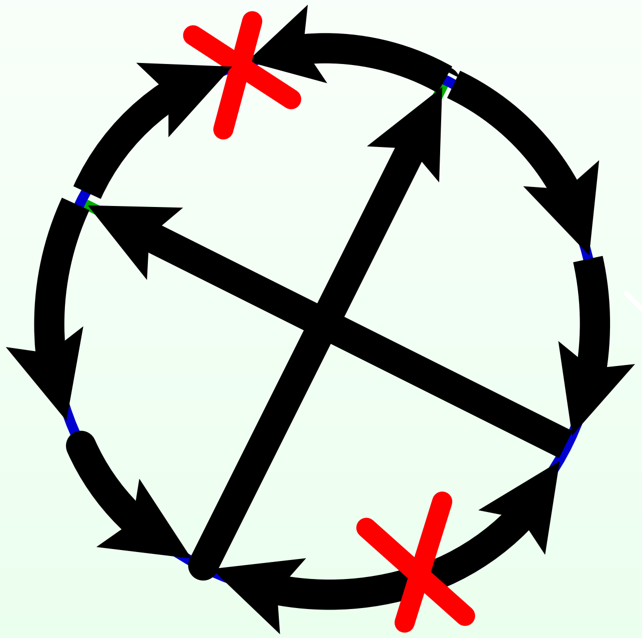

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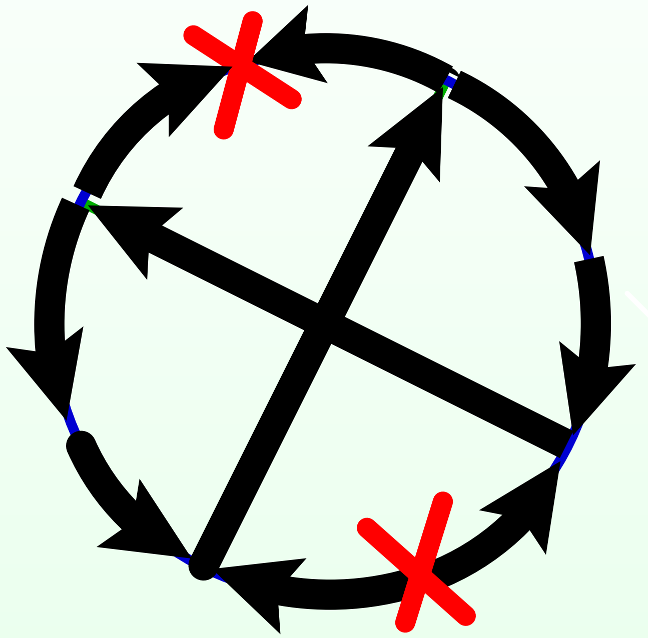

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Orient the complement of the 0-cycle realizing it,
to get *vice-orientation* of the chord diagram.

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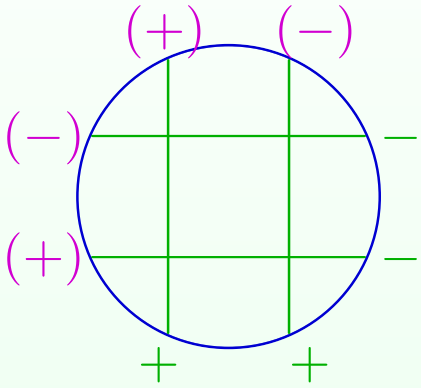

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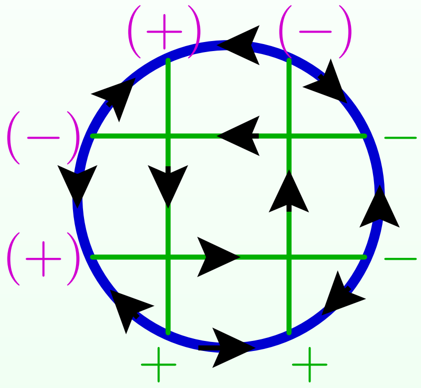

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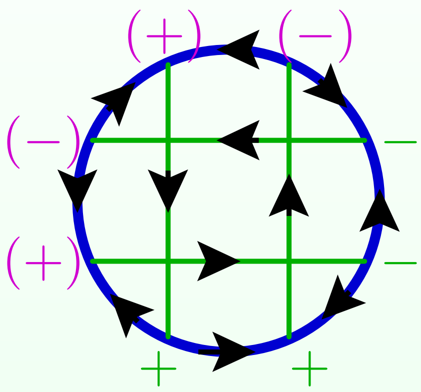

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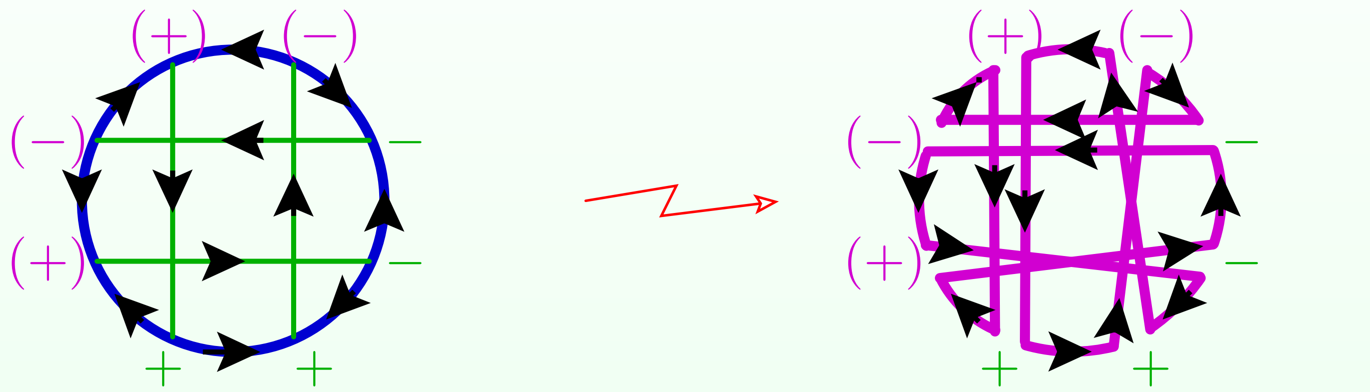

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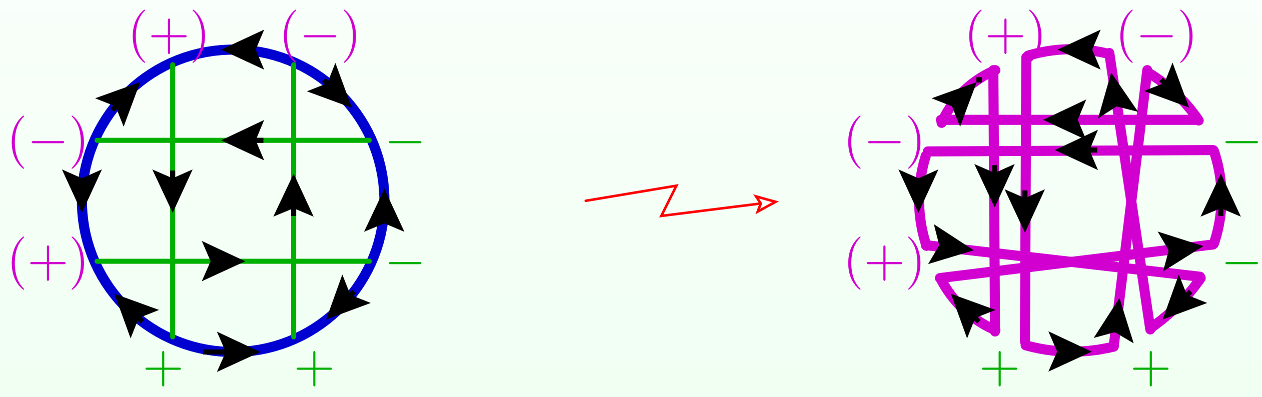

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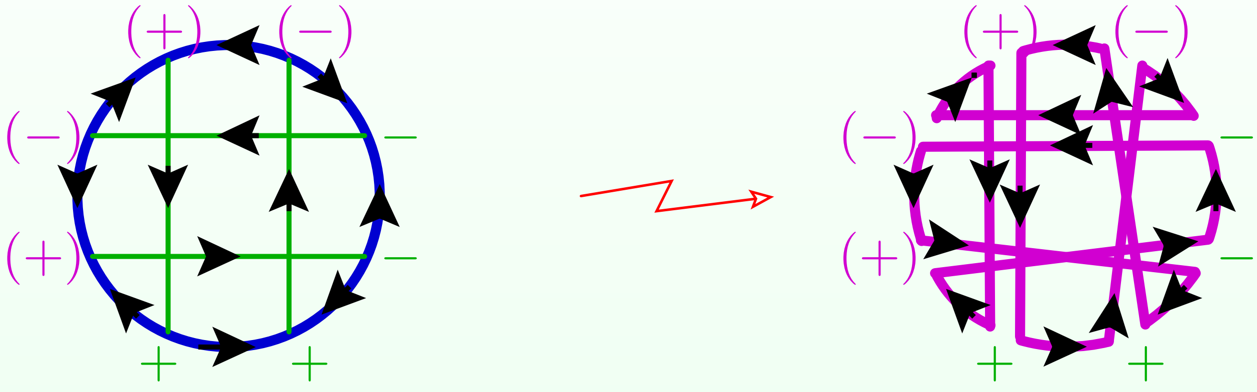
Similarly, a vice-orientation of a signed chord diagram induces a vice-orientation of each result of its smoothing.

Orientation of a smoothened chord diagram

If a chord diagram is oriented,



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Similarly, a vice-orientation of a signed chord diagram induces a vice-orientation of each result of its smoothing.

Theorem (Manturov, Viro) *Definition of the Khovanov complex extended straightforwardly to an oriented framed chord diagram gives a complex invariant under Reidemeister moves preserving the orientation.*

Khovanov complex of framed chord diagram

Structure used in the construction

1. Framed chord diagram.

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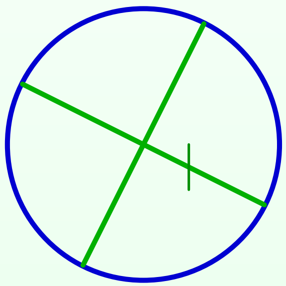

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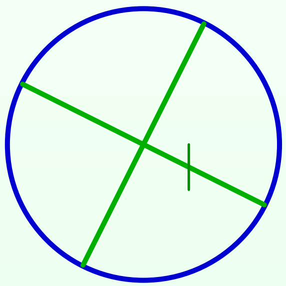

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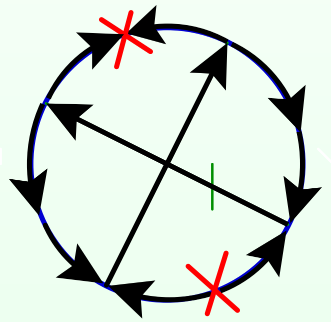

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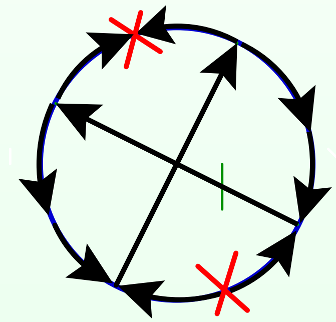
2. Vice-orientation of the chord diagram.
3. At each chord one of two arcs adjacent to its arrowhead is marked.



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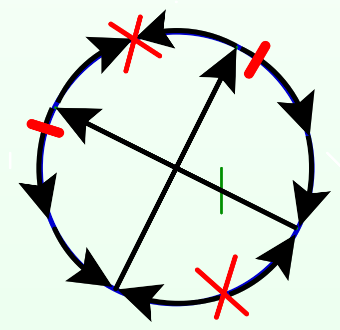

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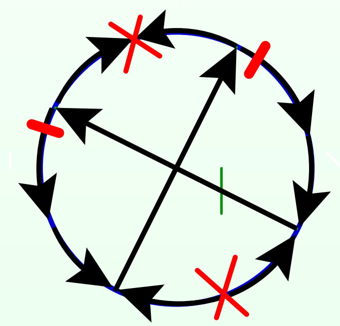

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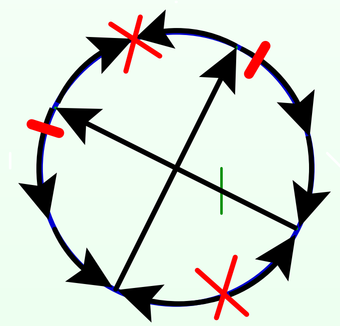

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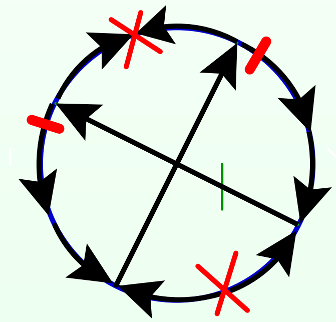

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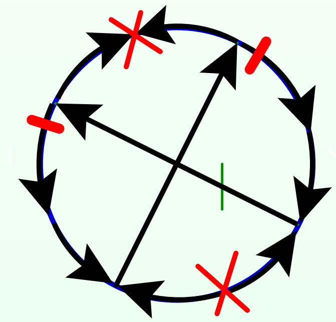

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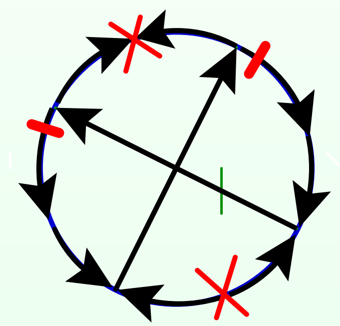

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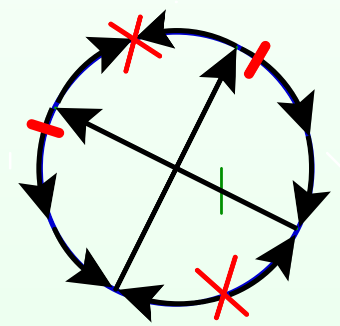

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Homology does not depend on the structure.

Involution in the Frobenius algebra

Remind that $\mathcal{A}$ is a Frobenius algebra generated by 1 and X with $X^2 = 0$.
with Grading: $\deg(1) = 0$, $\deg(X) = 2$.
and Comultiplication:
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Permutations of the components corresponds to the permutation isomorphism of the tensor product multiplied by the sign of the permutation.

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● On the second place put the other component passes though c (if there is one).

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Orient the first components according to the vice orientation at c. In these representations of V_s and V_t, define the map by multiplication or co-multipication.