Boundary Value Khovanov Homology

Oleg Viro

October 31, 2009

Khovanov homology

Kauffman bracket

$$\langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}]$$

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(a Laurent polynomial in A with integer coefficients).

Kauffman bracket

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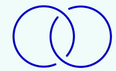
$\langle \text{Hopf link} \rangle =$ ⟨  ⟩

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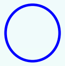
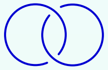
$\langle \text{unknot} \rangle =$ $\langle$  $\rangle$ $= -A^2 - A^{-2}$

$\langle \text{Hopf link} \rangle =$ $\langle$  $\rangle$ $= A^6 + A^2 + A^{-2} + A^{-6}$

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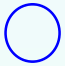
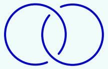
$\langle \text{unknot} \rangle =$ $\langle$  $\rangle$ $= -A^2 - A^{-2}$

$\langle \text{Hopf link} \rangle =$ $\langle$  $\rangle$ $= A^6 + A^2 + A^{-2} + A^{-6}$

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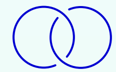
$\langle \text{Hopf link} \rangle =$ $\langle$  $\rangle$ $= A^6 + A^2 + A^{-2} + A^{-6}$

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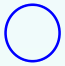
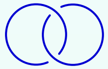
$\langle \text{unknot} \rangle =$ ⟨  ⟩ $= -A^2 - A^{-2}$

$\langle \text{Hopf link} \rangle =$ ⟨  ⟩ $= A^6 + A^2 + A^{-2} + A^{-6}$

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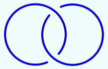
$\langle$Hopf link$\rangle$ = $\langle$  $\rangle$ $= A^6 + A^2 + A^{-2} + A^{-6}$

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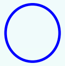
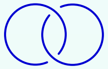
$\langle$unknot$\rangle$ = $\langle$  $\rangle$ $= -A^2 - A^{-2}$

$\langle$Hopf link$\rangle$ = $\langle$  $\rangle$ $= A^6 + A^2 + A^{-2} + A^{-6}$

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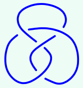
$\langle$figure-eight knot$\rangle$ = $\langle$  $\rangle$

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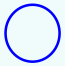
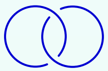
$\langle \text{unknot} \rangle =$ $\langle$  $\rangle$ $= -A^2 - A^{-2}$

$\langle \text{Hopf link} \rangle =$ $\langle$  $\rangle$ $= A^6 + A^2 + A^{-2} + A^{-6}$

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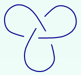
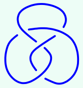
$\langle \text{trefoil} \rangle =$ $\langle$  $\rangle$ $= A^7 + A^3 + A^{-1} - A^{-9}$

$\langle \text{figure-eight knot} \rangle =$ $\langle$  $\rangle$ $= -A^{10} - A^{-10}$

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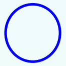
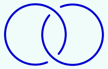
$\langle \text{unknot} \rangle =$ $\langle$  $\rangle$ $= -A^2 - A^{-2}$

$\langle \text{Hopf link} \rangle =$ $\langle$  $\rangle$ $= A^6 + A^2 + A^{-2} + A^{-6}$

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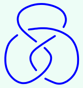
$\langle \text{figure-eight knot} \rangle =$ $\langle$  $\rangle$ $= -A^{10} - A^{-10}$

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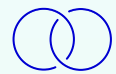
$\langle \text{Hopf link} \rangle = \langle$  $\rangle = A^6 + A^2 + A^{-2} + A^{-6}$

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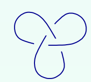
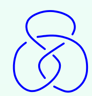
$\langle \text{trefoil} \rangle = \langle$  $\rangle = A^7 + A^3 + A^{-1} - A^{-9}$

$\langle \text{figure-eight knot} \rangle = \langle$  $\rangle = -A^{10} - A^{-10}$

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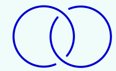
$\langle \text{Hopf link} \rangle = \langle$  $\rangle = A^6 + A^2 + A^{-2} + A^{-6}$

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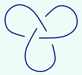
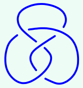
$\langle \text{trefoil} \rangle = \langle$  $\rangle = A^7 + A^3 + A^{-1} - A^{-9}$

$\langle \text{figure-eight knot} \rangle = \langle$  $\rangle = -A^{10} - A^{-10}$

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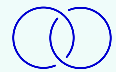
$\langle$Hopf link$\rangle$ = $\langle$  $\rangle = A^6 + A^2 + A^{-2} + A^{-6}$

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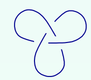
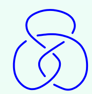
$\langle$trefoil$\rangle$ = $\langle$  $\rangle = A^7 + A^3 + A^{-1} - A^{-9}$

$\langle$figure-eight knot$\rangle$ = $\langle$  $\rangle = -A^{10} - A^{-10}$

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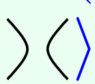
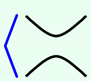
3. $\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$ (*Kauffman Skein Relation*) .

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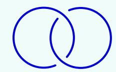
$\langle \text{Hopf link} \rangle = \langle$  $\rangle = A^6 + A^2 + A^{-2} + A^{-6}$

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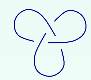
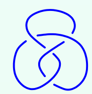
$\langle \text{trefoil} \rangle = \langle$  $\rangle = A^7 + A^3 + A^{-1} - A^{-9}$

$\langle \text{figure-eight knot} \rangle = \langle$  $\rangle = -A^{10} - A^{-10}$

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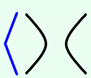
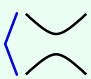
3. $\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$ (*Kauffman Skein Relation*) .

Uniqueness is obvious.

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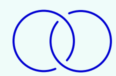
$\langle \text{Hopf link} \rangle = \langle$  $\rangle = A^6 + A^2 + A^{-2} + A^{-6}$

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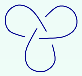
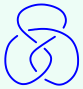
$\langle \text{trefoil} \rangle = \langle$  $\rangle = A^7 + A^3 + A^{-1} - A^{-9}$

$\langle \text{figure-eight knot} \rangle = \langle$  $\rangle = -A^{10} - A^{-10}$

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3. $\langle \times \rangle = A \langle \rangle\langle \rangle + A^{-1} \langle \asymp \rangle$ (*Kauffman Skein Relation*) .

Uniqueness is obvious. Invariant under Reidemeister 2 and 3.

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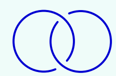
$\langle \text{Hopf link} \rangle = \langle$  $\rangle = A^6 + A^2 + A^{-2} + A^{-6}$

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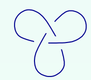
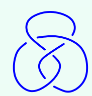
$\langle \text{trefoil} \rangle = \langle$  $\rangle = A^7 + A^3 + A^{-1} - A^{-9}$

$\langle \text{figure-eight knot} \rangle = \langle$  $\rangle = -A^{10} - A^{-10}$

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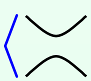
3. $\langle$  $\rangle = A \langle \,)(\, \rangle + A^{-1} \langle$  $\rangle$ (*Kauffman Skein Relation*) .

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Under Reidemeister 1 it multiplies by $-A^{\pm 3}$.

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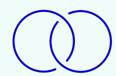
$\langle$Hopf link$\rangle =$ $\langle$  $\rangle = A^6 + A^2 + A^{-2} + A^{-6}$

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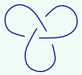
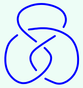
$\langle$trefoil$\rangle =$ $\langle$  $\rangle = A^7 + A^3 + A^{-1} - A^{-9}$

$\langle$figure-eight knot$\rangle =$ $\langle$  $\rangle = -A^{10} - A^{-10}$

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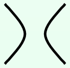
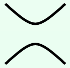
3. $\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$ (*Kauffman Skein Relation*) .

Uniqueness is obvious. Invariant under Reidemeister 2 and 3.

Under Reidemeister 1 it multiplies by $-A^{\pm 3}$.

$$(-A)^{-3w(D)}\langle D \rangle = \text{Jones}_D(-A^2)$$

Kauffman state sum

A state of diagram is a distribution of markers over all crossings.

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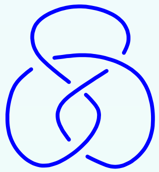
Knot diagram: 

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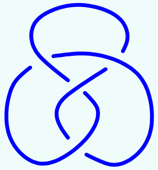
Knot diagram:  and its states:

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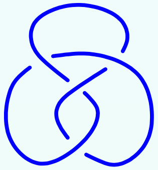
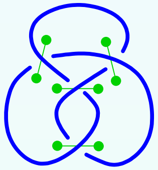
Knot diagram:  and its states:  ,

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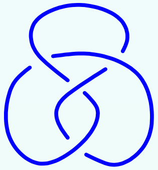
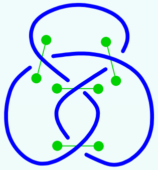
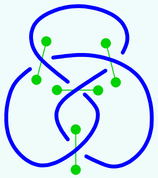
Knot diagram:  and its states: , ,

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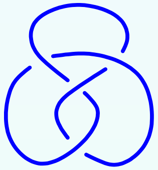
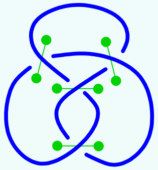
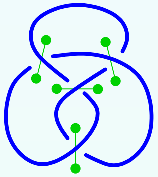
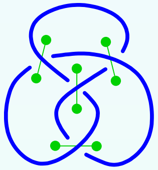
Knot diagram:  and its states: , , ,

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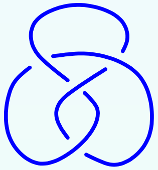
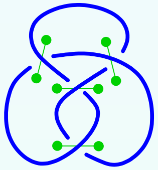
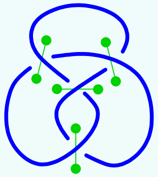
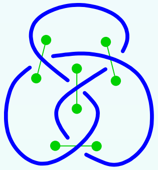
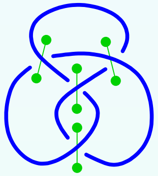
Knot diagram:  and its states: , , , ,

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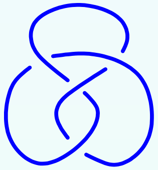
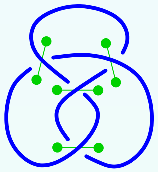
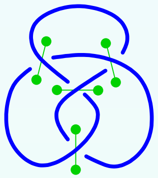
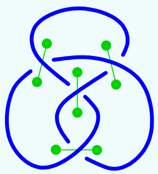
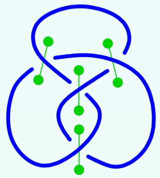
Knot diagram:  and its states: , , , , ...

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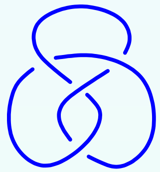
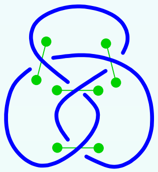
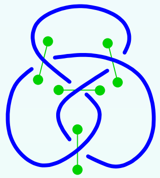
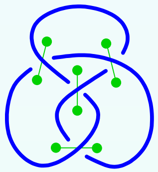
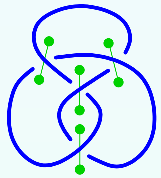
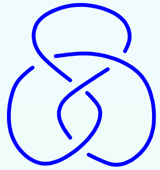
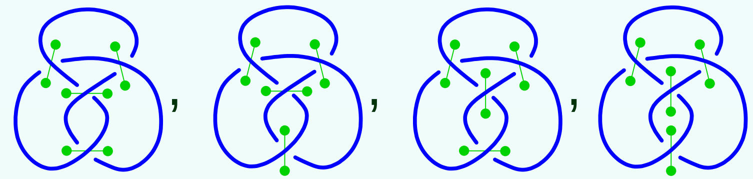
Knot diagram:  and its states: , , , , . . .

Totally 2^c states, where c is the number of crossings.

Kauffman state sum

A state of diagram is a distribution of markers over all crossings.

Knot diagram:  and its states: , . . .

Totally 2^c states, where c is the number of crossings.

Three numbers associated to a state s:

Kauffman state sum

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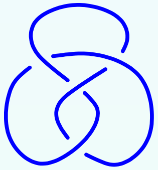
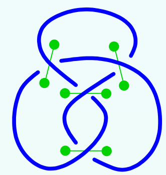
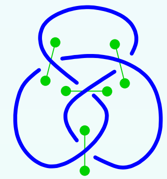
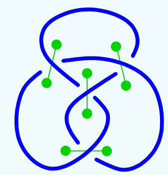
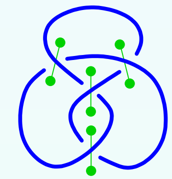
Knot diagram:  and its states: , , , , ...

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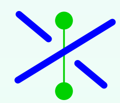
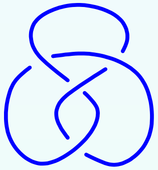
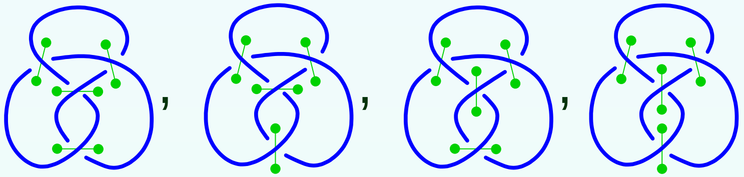
Three numbers associated to a state s:

1. the number $a(s)$ of *positive* markers ,

Kauffman state sum

A state of diagram is a distribution of markers over all crossings.

Knot diagram:  and its states: , . . .

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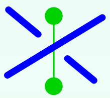
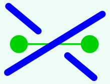
1. the number $a(s)$ of *positive* markers ,
2. the number $b(s)$ of *negative* markers ,

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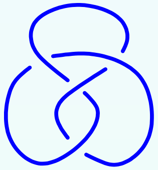
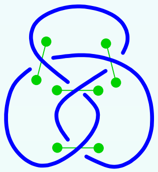
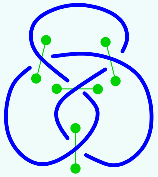
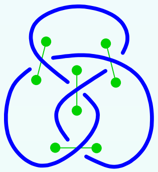
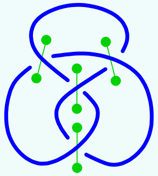
Knot diagram:  and its states: , , , , . . .

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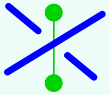
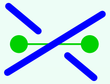
1. the number $a(s)$ of *positive* markers ,
2. the number $b(s)$ of *negative* markers ,
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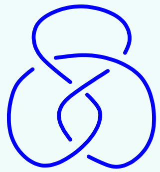
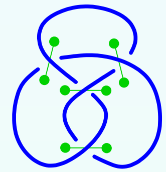
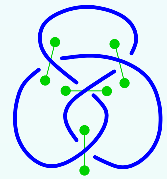
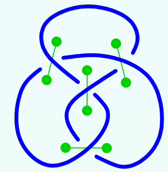
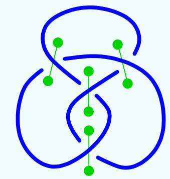
Knot diagram:  and its states: , , , , ...

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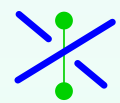
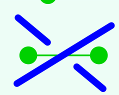
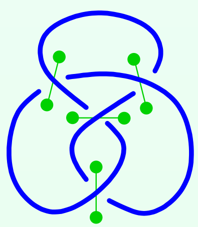
1. the number $a(s)$ of *positive* markers ,
2. the number $b(s)$ of *negative* markers ,
3. the number $|s|$ of components of the curve D_s obtained by smoothing along the markers:

$s =$  $\mapsto$

Kauffman state sum

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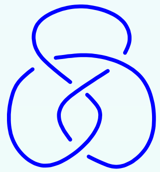
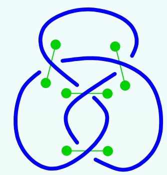
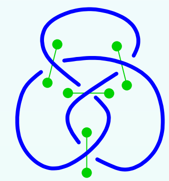
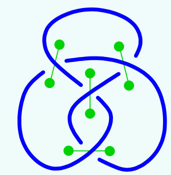
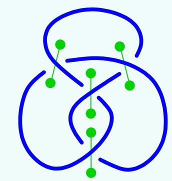
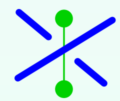
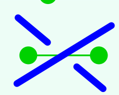
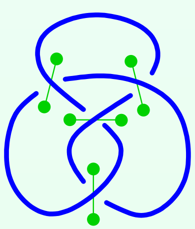
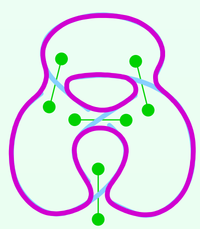
Knot diagram:  and its states: , , , , ...

Totally 2^c states, where c is the number of crossings.

Three numbers associated to a state s:

1. the number $a(s)$ of *positive* markers ,
2. the number $b(s)$ of *negative* markers ,
3. the number $|s|$ of components of the curve D_s obtained by smoothing along the markers:

$s =$  $\mapsto$ $D_s =$ 

Kauffman state sum

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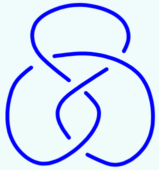
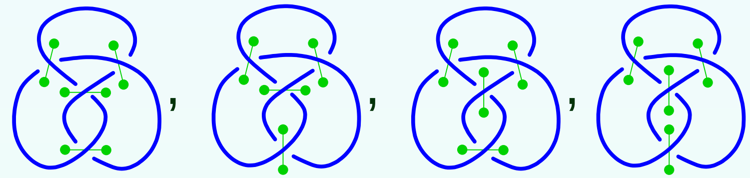
Knot diagram:  and its states: , . . .

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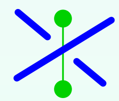
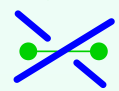
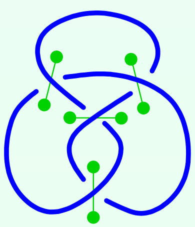
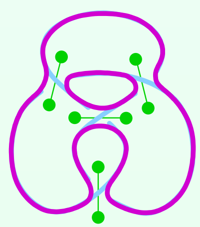
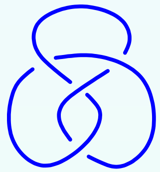
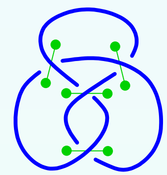
1. the number $a(s)$ of *positive* markers ,
2. the number $b(s)$ of *negative* markers ,
3. the number $|s|$ of components of the curve D_s obtained by smoothing along the markers:

$s =$  $\mapsto$ $D_s =$  $|s| = 2$

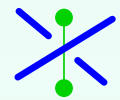
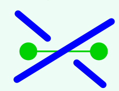
Kauffman state sum

A state of diagram is a distribution of markers over all crossings.

Knot diagram:  and its states: , . . .

Totally 2^c states, where c is the number of crossings.

Three numbers associated to a state s:

1. the number $a(s)$ of *positive* markers ,
2. the number $b(s)$ of *negative* markers ,
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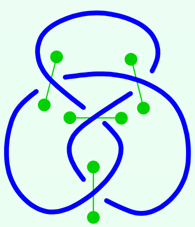
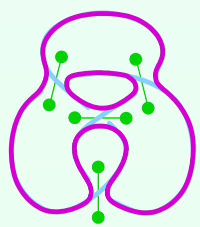
$s =$  $\mapsto \quad D_s =$  $\quad |s| = 2$

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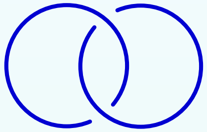
Example

Hopf link, 

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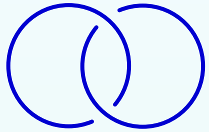
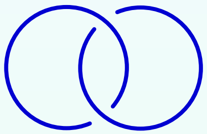
Hopf link, 

$\Big\langle$  $\Big\rangle =$

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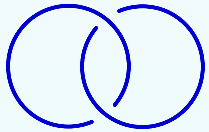
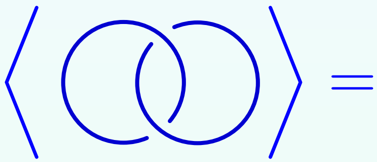
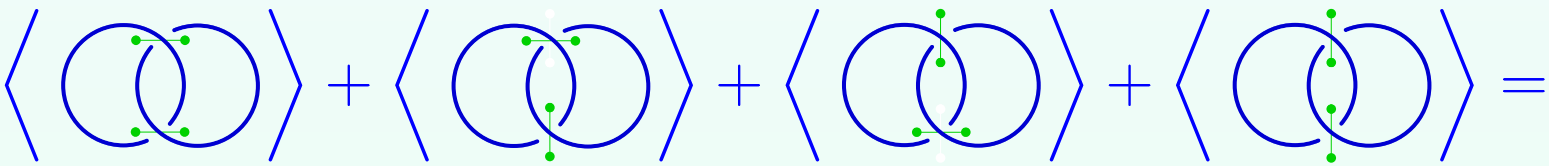
Hopf link, 





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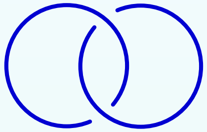
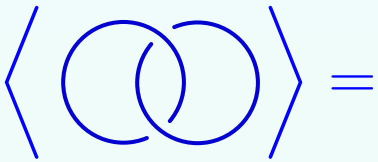
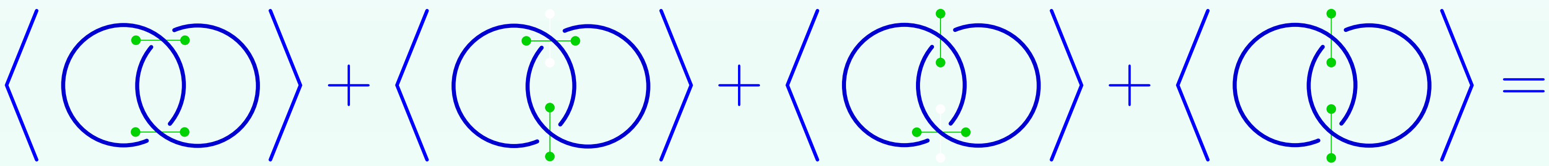
Hopf link, 





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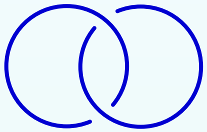
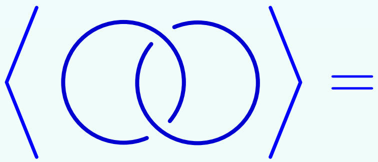
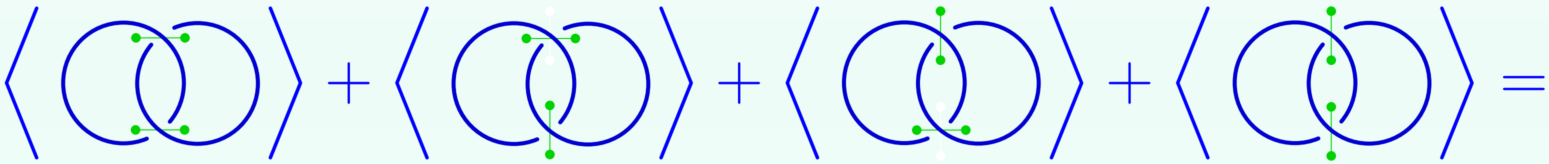
Hopf link, 





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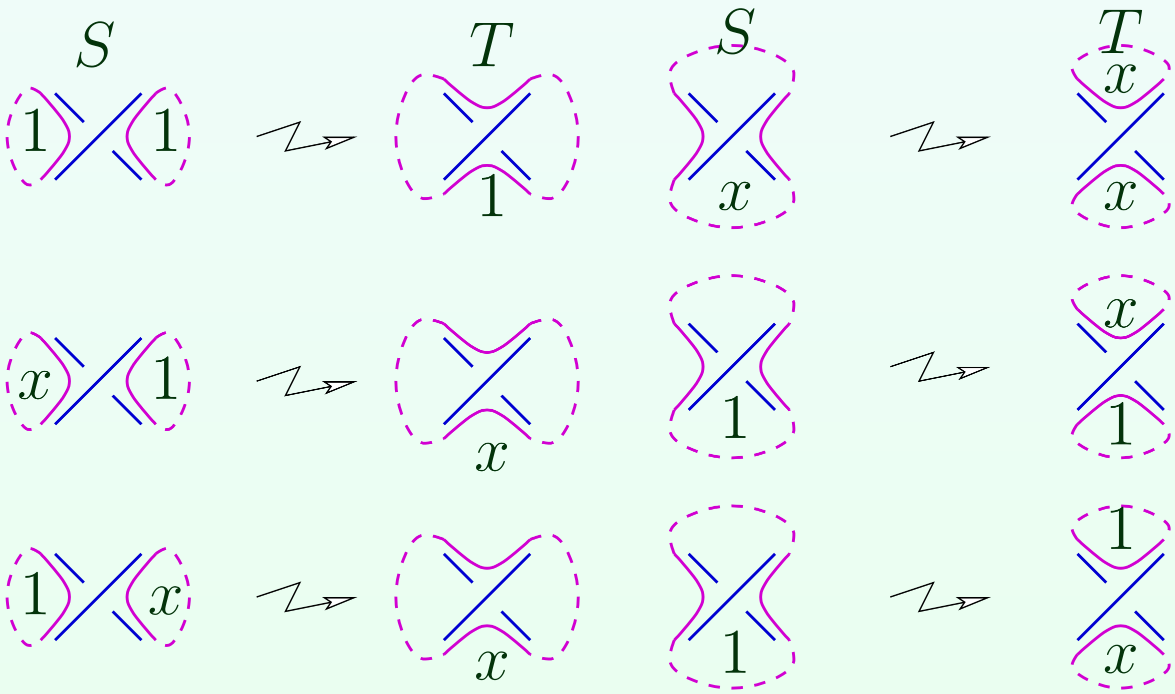

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Khovanov homology of tangles

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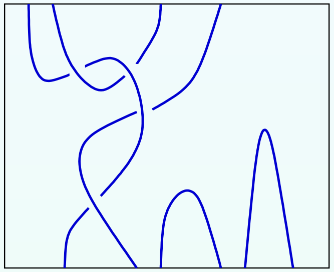

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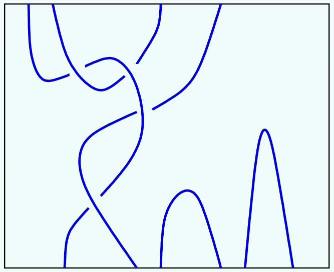

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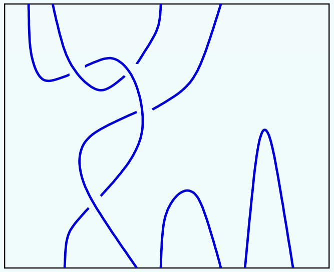

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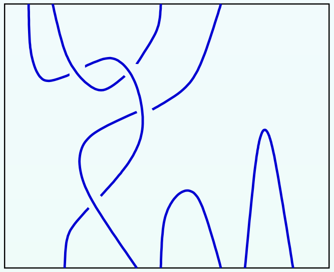

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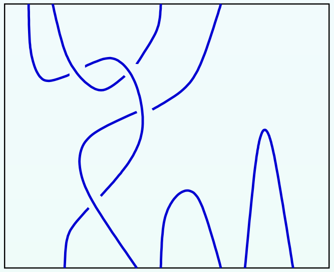

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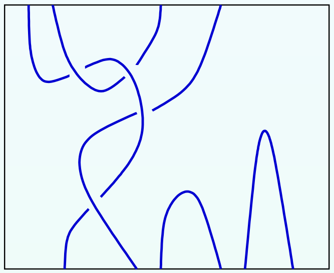

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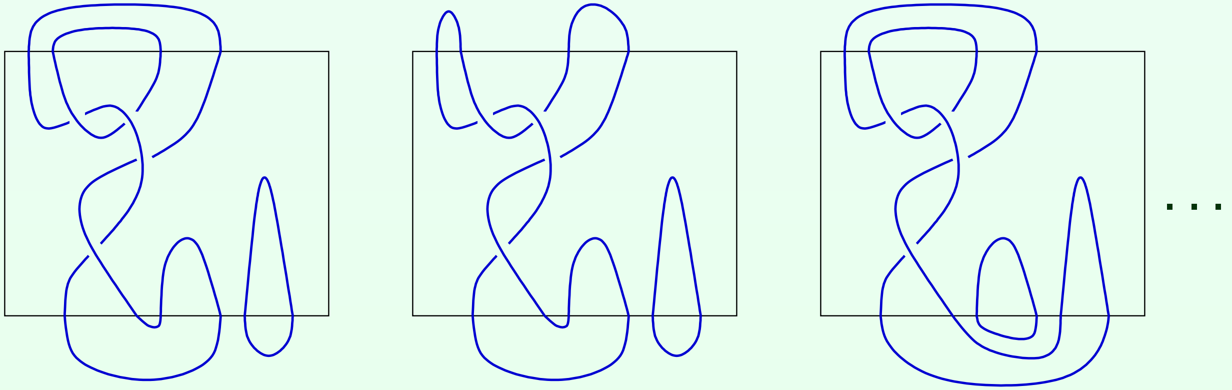

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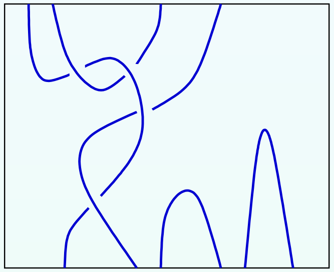

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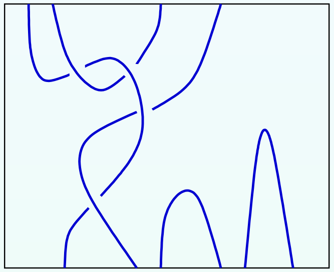

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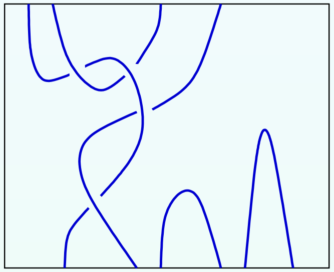

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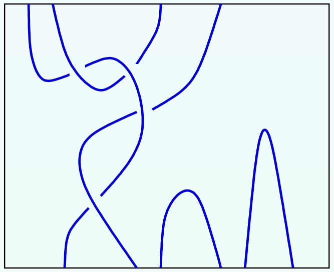

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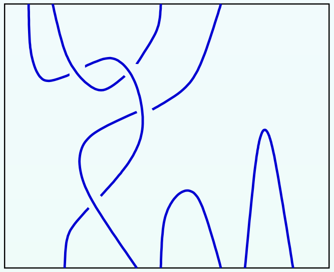

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No direct relation to the Reshetikhin-Turaev functor!

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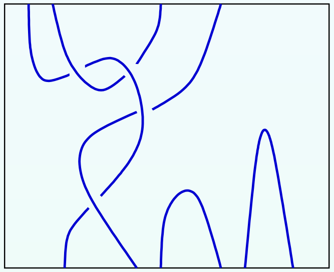

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The functoriality preserved.

Orientations replace generators

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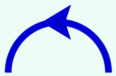	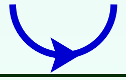	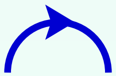	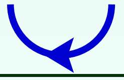
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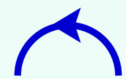	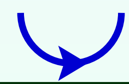	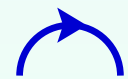	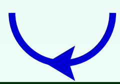
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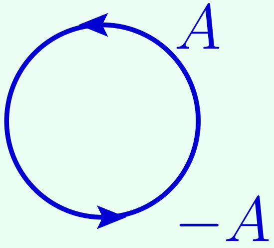
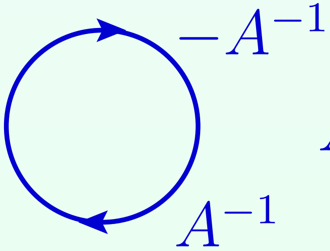


$A(-A) = -A^2$



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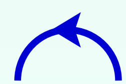	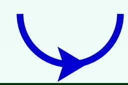	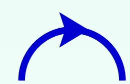	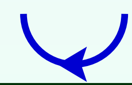
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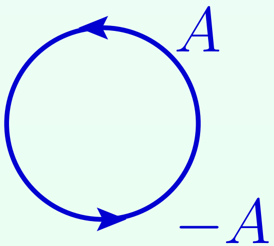

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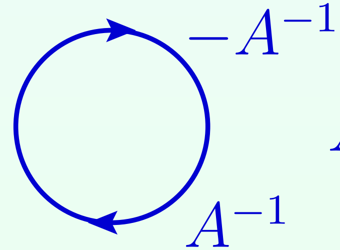

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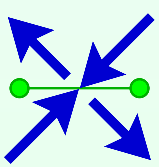
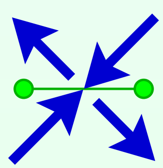
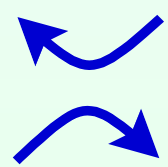
For  :

 $= A^{-1}$  A^{-1} $-A^{-1}$ $= -A^{-3}$

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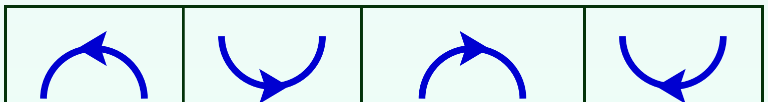

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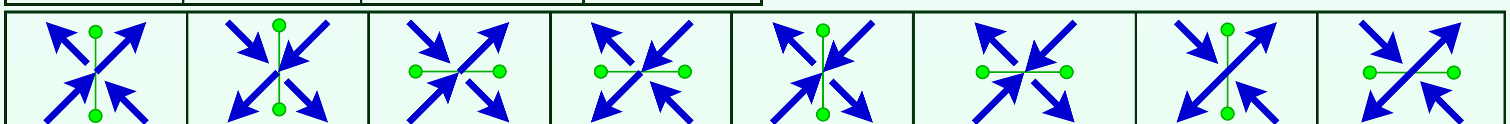
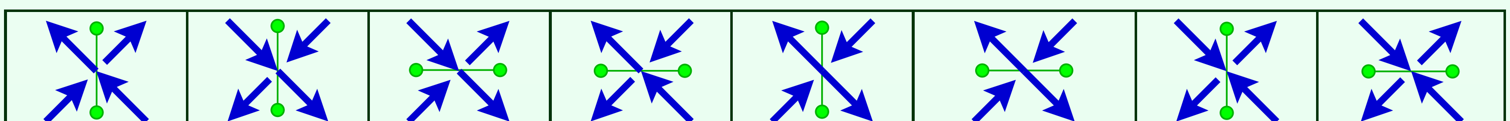


A	A	A^{-1}	A^{-1}	A	$-A^{-3}$	A	$-A$



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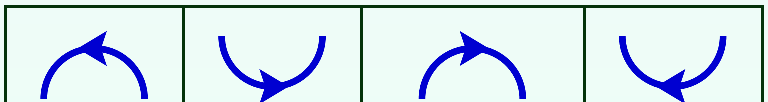

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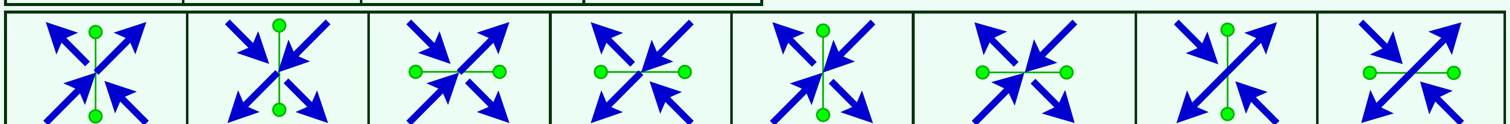

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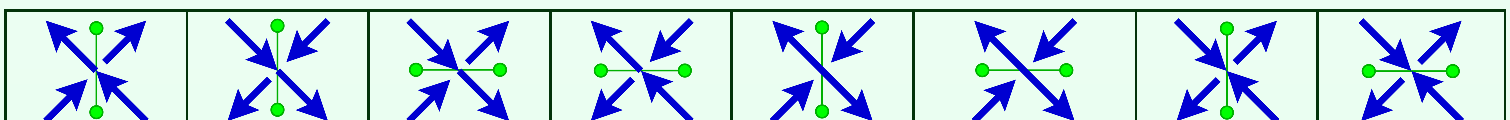

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The Kauffman state sum turns into the R-matrix state sum.

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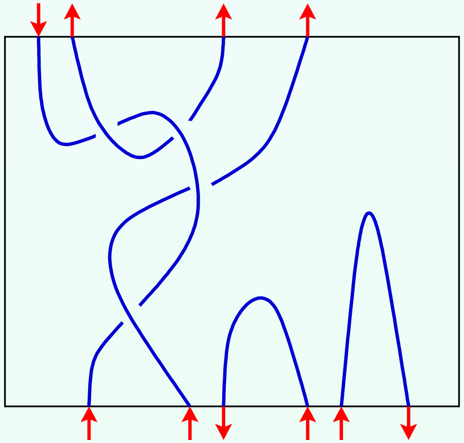

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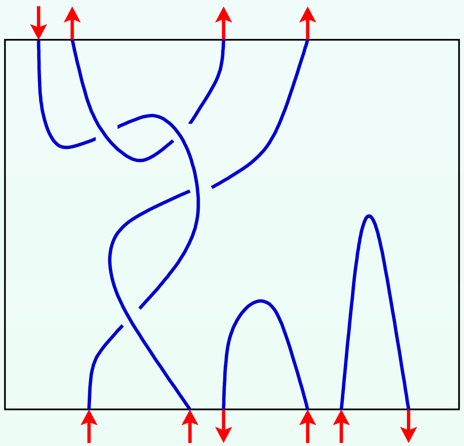

The matrix element can be computed as a state sum over distributions of markers at crossings and orientations of the corresponding smoothing.

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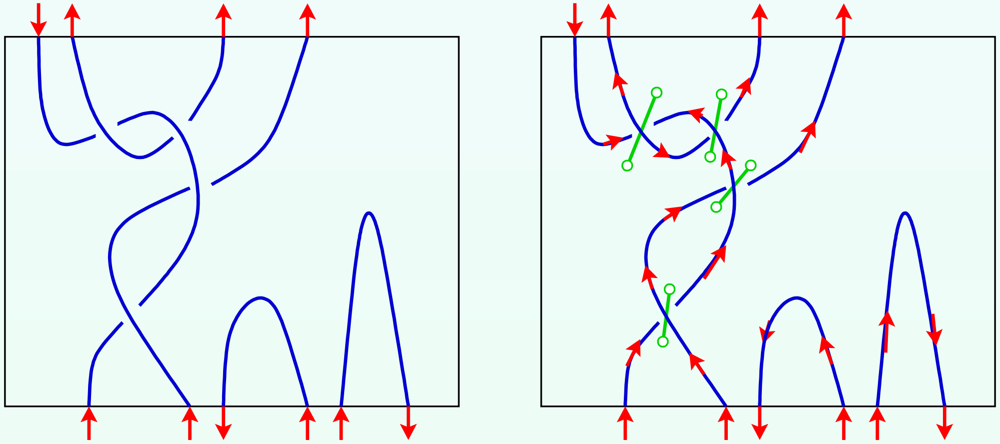

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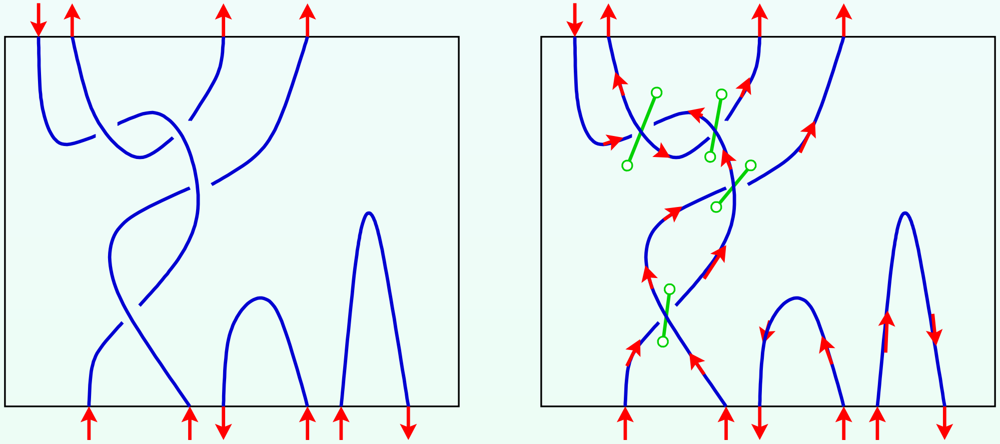

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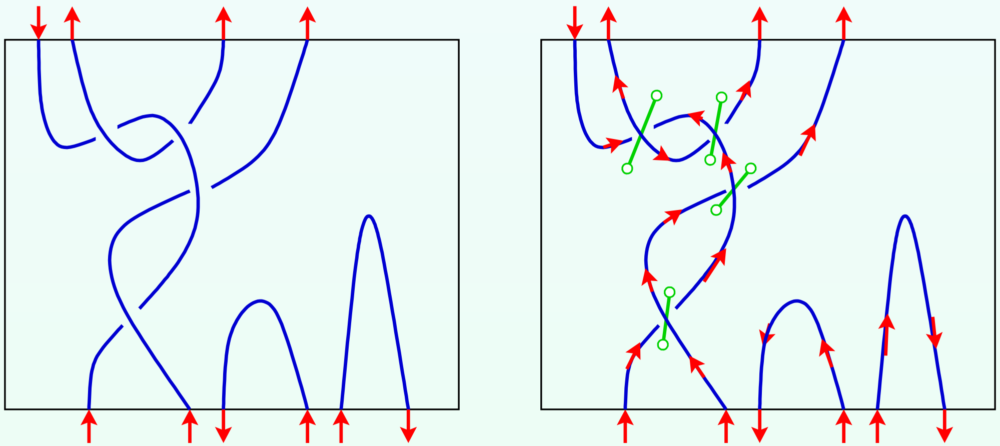

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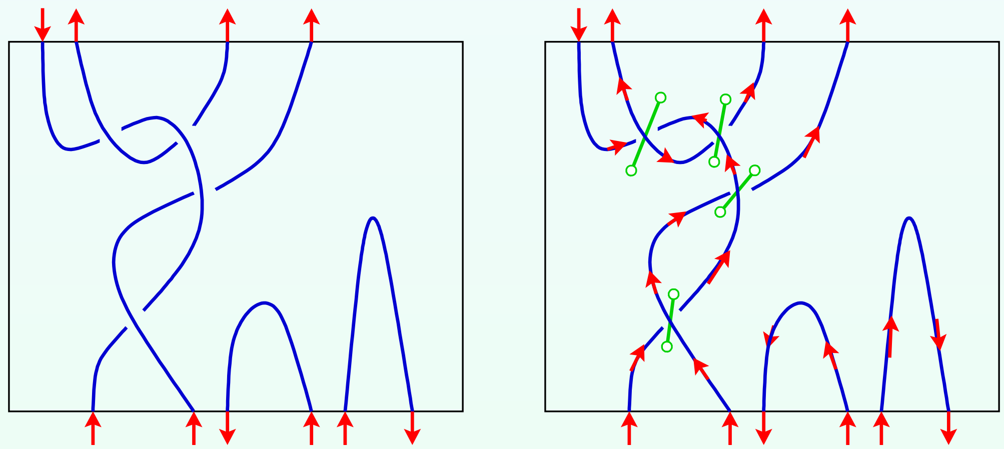

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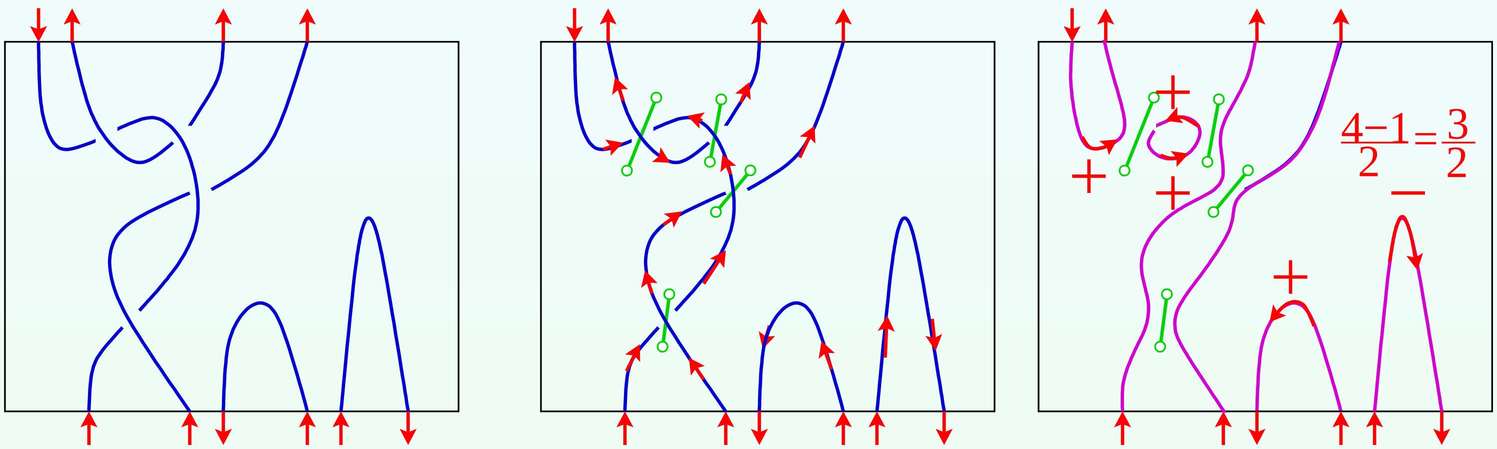

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Theorem 1. $d^2 = 0$

Theorem 2. An isotopy of a tangle defines homotopy equivalence of the chain complexes.

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