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PROBLEM CONTRIBUTION

Space of Smooth 1-Knots in a 4-Manifold: Is Its Algebraic Topology Sensitive to Smooth Structures?

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Abstract We discuss a possibility to get an invariant of a smooth structure on a closed simply connected 4-manifold from homotopy invariants of the space of loops smoothly embedded into the manifold.

Keywords Exotic smooth structures · 4-manifolds · Seiberg–Witten invariants · Alexander polynomial · Vassiliev invariants

1 Introduction

1.1 Diffeomorphic Versus Homeomorphic in High Dimensions

There are smooth manifolds, which are homeomorphic, but not diffeomorphic. It happens to manifolds of dimension >3. This phenomenon was discovered by Milnor (1956) for the 7-dimensional sphere S^7 in the fifties. A technique for classification of smooth structures on a manifold of dimension >4 was developed in the sixties. Homeomorphic, but not diffeomorphic manifolds were found in all dimensions >4, see Kervaire and Milnor (1963), Barden (1965) and Siebenmann (1971). A substantial part of the technique used in these works is not applicable in dimension 4.

1.2 Closed Simply Connected 4-Manifolds

In the eighties Freedman (1982) gave a topological classification of closed simply connected 4-manifolds and Donaldson (1987) proved that some smooth closed simply connected 4-manifolds, which are homeomorphic according to Freedman's results, are not diffeomorphic.

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For smooth closed simply connected 4-manifolds, the only homotopy invariant is the intersection form in the second homology with integer coefficients: two manifolds of this kind are homotopy equivalent iff their intersection forms are isomorphic (Whitehead 1949; Pontryagin 1949).

As Freedman (1982) proved, for manifolds of this type, homeomorphism is equivalent to homotopy equivalence. Thus, two smooth closed simply connected 4-manifolds are homeomorphic iff their intersection forms are isomorphic.

In contrast, on many closed simply connected 4-manifolds there are infinitely many smooth structures. Of course, the number of smooth structures on a closed 4-manifold cannot be uncountable (on a non-compact 4-manifold it can, as is the case for $\mathbb{R}^4$). There is no conjectural classification scheme for smooth structures on any closed 4-manifold. Stern (2006) entitled his survey paper by a question "Will we ever classify simply-connected smooth 4-manifolds?". In the introduction to his paper he wrote: "The subject is rich in examples that demonstrate a wide variety of disparate phenomena. Yet it is precisely this richness which, at the time of these lectures, gives us little hope to even conjecture a classification scheme."

1.3 Invariants Distinguishing Smooth Structures

Proving that homeomorphic manifolds of dimension 4 are not diffeomorphic required absolutely new tools. All the invariants, that have been used so far for proving that some homeomorphic simply connected closed smooth 4-manifolds are not diffeomorphic, are deeply rooted in analysis. They are based on counting solutions of some nonlinear partial differential equations. Most of the results on non-existence of diffeomorphisms between homeomorphic simply connected 4-manifolds have been obtained using the Donaldson and Seiberg–Witten invariants.

It is a long term challenge for topologists to find invariants which would distinguish smooth structures on a 4-manifold, but would be independent on analytic tools. This is not just an aesthetic issue: the analytic tools are not convenient in some situation.

For example, in dimension 4 smooth structures are closely related to PL-structures (piecewise linear structures). In any dimension a smooth structure on a manifold determines on it a specific PL-structure uniquely defined up to PL-homeomorphism. In dimensions ≤ 6 any PL-structure can be obtained in this way from a smooth structure and the smooth structure is unique up to diffeomorphism, see Siebenmann (1971). Hence any invariant of a smooth structure on a 4-manifold depends only on the PL-structure. However, in order to prove that two homeomorphic PL-manifolds of dimension 4 are not PL-homeomorphic, now one has to equip them with smooth structures, and then calculate the invariants. The techniques for calculation are not easy, especially if the smooth structure is not defined naturally (say, as the underlying structure on a complex surface), but just obtained by smoothing of a PL-structure.

It is worth mentioning a partial success of efforts towards eliminating of analysis. An invariant of smooth 4-manifolds, the Ozsváth and Szabó (2006) mixed invariant, conjecturally coinciding with the Seiberg–Witten invariant, has been redefined (Manolescu et al. 2009) in purely combinatorial terms. However, on one hand, it is difficult to calculate even for comparatively simple 4-manifolds, so it not an effective tool for distinguishing smooth structures, on the other hand, its combinatorial description has not clarified its nature, has not related it to the rest of topology.

All the known invariants distinguishing smooth structures on closed simply connected 4-manifold X require non-trivial intersection form on $H_2(X)$. In particular, for the 4-sphere these invariants give nothing, they cannot help in disproving of the 4-dimensional differential Poincaré conjecture, according to which any closed 4-dimensional manifold homeomorphic to S^4 is diffeomorphic to S^4.

Thus, a development of new approaches to building invariants of smooth structures is desirable.

2 Seiberg–Witten Versus Alexander

One of the most powerful invariants, which distinguish smooth structures on a 4-manifold, is the Seiberg–Witten invariant. It can be presented in several forms. In particular, the Seiberg–Witten invariant of a smooth closed simply-connected 4-manifold X can be identified with an element SW_X of $\mathbb{Z}[H_2(X)]$, the integer group algebra of the *second* homology group $H_2(X)$.

This identification makes the Seiberg–Witten invariant resembling the Alexander polynomial of a link. For a classical link $L \subset S^3$, the Alexander polynomial is an element of $\mathbb{Z}[H_1(S^3 \setminus L)]$, the integer group algebra of the *first* homology group.

The Alexander polynomial can be defined in many ways. In particular, it admits purely topological definitions.

The similarity between Seiberg–Witten invariant and Alexander polynomial gives a hope to find either a definition of Seiberg–Witten invariant free of heavy analysis, or, at least, to invent similar invariants that solve the same problems, but are defined and calculated in ways more traditional for topology. This hope is supported by a relation between the Seiberg–Witten invariant and the Alexander polynomial discovered by Fintushel and Stern (1998).

3 Fintushel–Stern Knot Surgery of a 4-Manifold

Let X be any simply connected closed smooth 4-manifold. Suppose that X contains a smoothly embedded torus T with simply connected complement $X \setminus T$ and with zero self-intersection number $T \circ T$. Since $T \circ T = 0$, the normal bundle of T is trivial, and a tubular neighborhood of T can be identified with $T \times D^2$.

A *knot surgery* on T takes away from X the interior of a tubular neighborhood $T \times D^2$ of T and attaches $S^1 \times (S^3 \setminus N_K)$ instead, where N_K is an open tubular neighborhood of a smooth knot $K \subset S^3$. Observe that the boundary of $S^3 \setminus N_K$ is diffeomorphic to the 2-torus $S^1 \times S^1$, and hence the boundary of the inserted piece $S^1 \times (S^3 \setminus N_K)$ is diffeomorphic to the 3-torus $S^1 \times S^1 \times S^1$ as well as the boundary $T \times \partial D^2$ of the piece removed. The attaching is performed by a diffeomorphism $S^1 \times \partial(S^3 \setminus N_K) \to T \times \partial D^2$ which maps $\{pt\} \times l$, where l is a longitude, i.e., a circle bounding in $S^3 \setminus K$, to a fiber $\{pt\} \times \partial D^2$. This requirement on the attaching map does not determine the map up to diffeotopy, and hence does not necessarily determine X_K up to diffeomorphism. However, by the following theorem, under some assumptions, all the manifolds obtained from the same (X, T) and $K \subset S^3$ by knot surgeries have the same Seiberg–Witten invariant.

Theorem 1 (Fintushel and Stern 1998) *Let X be a simply connected closed smooth 4-manifold with $b_+(X) > 1$, let X contain a smoothly embedded torus T with $T \circ T = 0$ and simply connected complement $X \smallsetminus T$. Let K be a knot in S^3. Then the result X_K of a knot surgery is homeomorphic to X and*

$$SW_{X_K} = SW_X \cdot \Delta_K(t^2),$$

where $t \in H_2(X)$ is the homology class realized by T and Δ_K is the symmetrized Alexander polynomial of K.

Present definitions of the Seiberg–Witten invariant are not applicable to $(S^3 \smallsetminus K) \times S^1$ or $(S^3 \smallsetminus N_K) \times S^1$. So, one cannot speak about $SW_{(S^3 \smallsetminus K)\times S^1}$. However, if the SW was extended to this setup and satisfied a reasonable additivity property, Fintushel–Stern Theorem would suggests that $SW_{(S^3 \smallsetminus K)\times S^1}$ should be Δ_K.

4 Is SW_X the Alexander Polynomial of Something?

The Alexander polynomial of a classical knot K is the order of $\mathbb{Z}[H_1(S^3 \smallsetminus K)]$-module $H_1(\widetilde{S^3 \smallsetminus K})$, where $\widetilde{S^3 \smallsetminus K} \to S^3 \smallsetminus K$ is the infinite cyclic covering. The automorphism group of this covering is $H_1(S^3 \smallsetminus K) = \mathbb{Z}$, it acts in the homology group $H_1(\widetilde{S^3 \smallsetminus K})$, turning it into a $\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$-module.

This module is called the *Alexander module* of K. It is finitely generated and admits a square matrix of relations. The determinant of this matrix is the Alexander polynomial. It is defined up to multiplication by units of $\mathbb{Z}[\mathbb{Z}]$ (that is by monomials $\pm t^k$).

According to this construction, the Alexander polynomial happens to belong to $\mathbb{Z}[H_1(S^3 \smallsetminus K)]$. The Seiberg–Witten invariant of X belongs to $\mathbb{Z}[H_2(X)]$. What could be a space Y such that $H_1(Y) = H_2(X)$?

5 The Loop Space ΩX?

There is an obvious candidate for such Y, the loop space ΩX of X. Indeed, $\pi_i(\Omega X) = \pi_{i+1}(X)$, and, in particular, $\pi_1(\Omega X) = \pi_2(X)$; in the case of simply connected X, $\pi_2(X)$ is isomorphic to $H_2(X)$ by the Hurewicz theorem. Thus $\pi_1(\Omega X) = H_2(X)$. Therefore $\pi_1(\Omega X)$ is commutative, and hence $H_1(\Omega X) = \pi_1(\Omega X) = H_2(X)$.

However, the loop space per se cannot do the job:

- First, the homotopy type of ΩX depends only on the homotopy type of X. Therefore homotopy invariants of ΩX cannot distinguish smooth structures on X.
- Second, ΩX is an H-space. Therefore $\pi_1(\Omega X)$ acts on $H_*(\widetilde{\Omega X})$ trivially.

6 The Knot Space?

How to improve ΩX?

First, let us make it closer to the smooth structure of X. The loop space ΩX contains the space $\Omega_{Diff} X$ of differentiable loops $S^1 \to X$ that have at the base point fixed non-vanishing differential. The replacement of ΩX by $\Omega_{Diff} X$ does not change the homotopy type: well-known approximation theorems imply that $\Omega_{Diff} X$ is a deformation retract of ΩX.

Let $KX \subset \Omega_{Diff} X$ be the subspace which consists of loops that are smooth embeddings. Denote $\Omega_{Diff} X \smallsetminus KX$ by DX. Observe that $\operatorname{codim}_{\Omega_{Diff} X} DX = 2$, hence the inclusion homomorphism $\pi_1(KX) \to \pi_1(\Omega_{Diff} X) = \pi_1(\Omega X)$ is onto.

Obvious suggestion: consider the covering $\widetilde{KX} \to KX$ induced by the universal covering $\widetilde{\Omega X} \to \Omega X$. The automorphism group of the covering $\widetilde{\Omega X} \to \Omega X$ is $H_1(\Omega X) = H_2(X)$. Therefore $H_*(\widetilde{\Omega X})$ and $H_*(\widetilde{KX})$ are modules over $\mathbb{Z}[H_1(\Omega X)] = \mathbb{Z}[H_2(X)]$.

The action of $H_1(\Omega X)$ in $H_*(\widetilde{\Omega X})$ is trivial, since ΩX is an H-space, while KX is not an H-space and $H_*(\widetilde{KX})$ may be an interesting $\mathbb{Z}[H_2(X)]$-module. It has a broad range of invariants belonging to $\mathbb{Z}[H_2(X)]$ similar to the Alexander polynomial. Indeed, if X is simply connected, then $H_2(X)$ is a free abelian group of finite rank r, and $\mathbb{Z}[H_2(X)]$ is isomorphic to the ring $\Lambda_r = \mathbb{Z}[t_1, t_1^{-1}, t_2, t_2^{-1}, \dots t_r, t_r^{-1}]$ of Laurent polynomial in r variables with integer coefficients, the same ring as in the situation of the Alexander module of a classical link.

A finitely generated module M over Λ_r gives rise to a filtration of $\mathbb{Z}[H_2(X)]$

$$\mathrm{Fitt}_0(M) \subset \mathrm{Fitt}_1(M) \subset \cdots \subset \mathbb{Z}[H_2(X)]$$

by *Fitting ideals*. The ith Fitting ideal $\mathrm{Fitt}_i(M)$ is generated by the minors (determinants of submatrices) of order $r - i$ of the matrix of defining relations for M In the topological literature Fitting ideals are called also *elementary ideals*. A generator of the minimal principal ideal containing $\mathrm{Fitt}_i(M)$ is denoted by $\Delta_i(M)$.

In the context of link theory, when $\Lambda_r = H_1(S^3 \smallsetminus L)$, where L is an r-component link and $M = H_1(\widetilde{S^3 \smallsetminus L})$ where $(\widetilde{S^3 \smallsetminus L})$ is the maximal abelian covering space of $S^3 \smallsetminus L$, the Laurent polynomial $\Delta_i(M)$ is called the ith Alexander polynomial of L. The 0th Alexander polynomial is one of the oldest link invariants. It was introduced by Alexander (1928).

Similarly, for a smooth simply connected closed 4-manifold X, each $H_i(\widetilde{KX})$, as a module over $\mathbb{Z}[H_2(X)]$, gives rise to a sequence of Laurent polynomials. They resemble the Seiberg–Witten invariant. At least, they belong to the same $\mathbb{Z}[H_2(X)]$. The modules $H_i(\widetilde{KX})$ are also invariants of X.

The problem: *Does there exist homeomorphic smooth simply connected closed 4-manifolds X_1, X_2 such that $H_*(\widetilde{KX_1})$ and $H_*(\widetilde{KX_2})$ are not isomorphic?*

7 Is this Plausible?

The author discussed the problem with many leading specialists in the field and asked them this question. The answers spread over a broad spectrum. Only one expert expressed a definitely negative opinion. On the other end, there were also very enthusiastic reactions.

The most convincing argument is that for different smooth structures on the same topological manifold X a topologically observable difference between smooth structures was the minimal genus of a smoothly embedded surface realizing an element of $H_2(X)$. For any smooth structure each class can be realized by an immersed sphere with transverse self-intersection (by the Hurewicz theorem and transversality), but the minimal number of double point of such immersion depends on the structure. It is greater than or equal to the minimal genus of a smoothly embedded surface realizing the class.

An immersion $f : S^2 \to X$ can be used to produce a loop in the space of loops $\Omega_{Diff} X$. A loop in $\Omega_{Diff} S^2$ sweeping the whole S^2 composed with f gives a loop in $\Omega_{Diff} X$. Generically this gives a loop in KX, because under a generic choice of loop in $\Omega_{Diff} S^2$, points in the preimage of a double point of f are passed at different moments. However, in families of loops realizing elements of high-dimensional homology groups $H_k(\Omega_{Diff} S^2) = \mathbb{Z}$ these pairs of points appear on some of the loops. Such a family does not realize a homology class in KX or $\widetilde{KX}$.

8 Vassiliev Invariants for 4-Manifolds?

How to calculate $H_i(\widetilde{KX})$? One may apply Vassiliev's (1994) idea, which led to discovery of the Vassiliev knot invariants: start the calculation with a study of the dual cohomology, the cohomology of the space of singular knots. The space of singular knots has a rich geometric structure.

The space DX consists of differentiable loops that are not embeddings. It fits to the collection of discriminant hypersurfaces studied by Vassiliev (1994). The universal covering $\widetilde{\Omega X} \to \Omega X$ defines a covering $\widetilde{DX} \to DX$.

Resolve singularities of the discriminant DX, as Vassiliev did. This gives rise to a filtration in $H^*(\widetilde{DX})$. The first terms of these filtration are easier to calculate than the whole cohomology group.

At first glance, the situation is much more complicated than in the original setup in the theory of Vassiliev knot invariants. Let us examine the extra difficulties.

First, instead of the space of singular knots, we have to deal with its covering space. What does happen, when we pass from knots to points of a covering space?

In general, if $p : X \to B$ is a covering, $b_0 \in B$, $x_0 \in X$ are points such that $p(x_0) = b_0$ and X is path connected, then a point $x \in X$ is uniquely determined by its image $p(x) \in B$, a path $s : I \to B$ such that $s(0) = b_0$, and its covering path $\widetilde{s} : I \to X$ starts at x_0 and finishes at x. The path s is defined up to path homotopy and multiplying by loops covered by loops in X.

In particular, a point of $\widetilde{\Omega X}$ (no matter if it belongs to $\widetilde{KX}$ or $\widetilde{DX}$) is defined by a loop $u : S^1 \to X$ and a continuous map $f : D^2 \to X$ with $f|_{S^1} = u$ considered up to homotopy which is fixed on S^1. The action of $H_2(X)$ in $\widetilde{\Omega X}$ is realized in this model as addition to the homotopy class of f homotopy classes of maps $S^2 \to X$ realizing elements of $H_2(X)$.

Second, the space of all loops in the Vassiliev setup is contractible (Recall that loops there have the base point and the tangent vector at the base point fixed). This allowed to apply the Alexander duality between the homology of the discriminant and cohomology of its complement, the space of knots. This was done in a finite dimensional (say, polynomial) approximation of the loop space.

Here the loop space is not contractible. Therefore, the Alexander duality is not applicable. However, it may be replaced by the Alexander–Poincaré duality between the homology of $\widetilde{KX}$ and a relative cohomology, the cohomology of $(\widetilde{\Omega_{Diff} X}, \widetilde{DX})$.

9 An Alternative Approach: String Topology

The homology of loop space $\Omega_{Diff} X$ and its universal covering accommodate a rich structure of the string topology operations (Chas and Sullivan 1999). Geometrically, one can expect that this structure incorporates the same information as the natural filtration of the discriminant and the Alexander–Poincaré duality. Apparently the connection have not been investigated. Nonetheless, it would be natural to expect that the string topology is as strong as the invariants discussed above.

Problem. *Is the string topology sensitive to smooth structures on 4-manifolds?*

It would be interesting also to find direct relations between the Vassiliev invariants theory and the string topology in the setup of classical knot theory.

References

Alexander, J.W.: Topological invariants of knots and links. Trans. Am. Math. Soc. **30**(2), 275–306 (1928)

Barden, D.: Simply connected five-manifolds. Ann. Math. **82**(2), 365–385 (1965)

Chas, M., Sullivan, D.: String topology. Preprint arXiv:math/9911159 (1999)

Donaldson, S.: Irrationality and the h-cobordism conjecture. J. Differ. Geom. **26**(1), 141–168 (1987)

Fintushel, R., Stern, R.: Knots, links, and 4-manifolds. Invent. Math. **134**(2), 363–400 (1998)

Freedman, M.H.: The topology of four-dimensional manifolds. J. Differ. Geom. **17**(3), 357–453 (1982)

Kervaire, M., Milnor, J.W.: Groups of homotopy spheres. Ann. Math. **77**, 504–537 (1963)

Manolescu, C., Ozsváth, P.S., Thurston, D.P.: Grid diagrams and Heegaard Floer invariants. arXiv:0910.0078 [math.GT] (2009)

Milnor, J.W.: On manifolds homeomorphic to the 7-sphere. Ann. Math. **64**, 399–405 (1956)

Ozsváth, P.S., Szabó, Z.: Holomorphic triangles and invariants for smooth four-manifold. Adv. Math. **202**(2), 326–400 (2006)

Pontryagin, L.S.: On classification of four-dimensional manifolds. Uspekhi Mat. Nauk **4**(4), 157–158 (1949). (Russian)

Stern, R.J.: Will we ever classify simply-connected smooth 4-manifolds? Clay Math. Proc. **5**, 225–239 (2006)

Siebenmann, L.C.: Topological manifolds. In: Proceedings of the international congress of mathematicians nice, vol. 2, September 1970, Gauthier-Villars, pp. 133–163, 1971

Vassiliev, V.A.: Complements of Discriminants of Smooth Maps: Topology and Applications. Translations of Mathematical Monographs Series, 2nd edn. American Mathematical Society, Providence (1994)

Whitehead, J.H.C.: On simply connected 4-dimensional polyhedra. Comment. Math. Helv. **22**, 48–92 (1949)