Complex orientations of real algebraic surfaces

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1 Introduction

In this paper we will deal with *real algebraic surfaces*. However let me start with some well known facts on real algebraic *curves,* to motivate the forthcoming exposition.

The set $\mathbb{R}A$ of real points of a non-singular real algebraic curve A lies in the set $\mathbb{C}A$ of its complex points. Topology of the placement of $\mathbb{R}A$ in $\mathbb{C}A$ can be described in terms of at most two very concise characteristics.

First, either $\mathbb{R}A$ divides $\mathbb{C}A$ into two halves, or $\mathbb{R}A$ does not divide $\mathbb{C}A$. In the first case the curve A is said to be of *type* I, and called *dividing curve,* in the second it is said to be of *type* II, and called *non-dividing curve.*

Second, if the curve is of type I, the canonical orientation of $\mathbb{C}A$ (determined by the complex structure of $\mathbb{C}A$) determines orientations of both of its halves and they in turn induce orientations on $\mathbb{R}A$, as on their common boundary. These orientations are opposite to each other. They are called *complex orientations* of A.

The type of curve enhanced, in the case of type I, by the complex orientations describes the inclusion $\mathbb{R}A \to \mathbb{C}A$ up to homeomorphism of $\mathbb{C}A$.

It was F. Klein who introduced (more than hundred years ago) the two types of real algebraic curves, see [11]. In seventies V. A. Rokhlin [15], [16] introduced the complex orientations. For the case of non-singular real plane projective curves Rokhlin found also relationships between placement of $\mathbb{R}A$ in $\mathbb{C}A$ and placement of the same set $\mathbb{R}A$ in the real projective plane $\mathbb{R}P^2$. These relationships proved to be very important in the subsequent development of topology of plane real algebraic curves.

A traditional viewpoint on problems of topology of real algebraic curves was that the main problem is to classify up to homeomorphisms of $\mathbb{R}P^2$ placements $\mathbb{R}A \subset \mathbb{R}P^2$ for non-singular real algebraic curves A of a given degree.

At first, the relationships found by Rokhlin seemed to be useless from this traditional viewpoint. They relate placement of $\mathbb{R}A$ in $\mathbb{C}A$ to placement of $\mathbb{R}A$ in $\mathbb{R}P^2$, but at that time (middle seventies) they gave nothing new on placement of $\mathbb{R}A$ in $\mathbb{R}P^2$ itself. However later, when additional restrictions were found (the most remarkable one was found by T. Fiedler [4]), they gave new restrictions on placement in $\mathbb{R}P^2$ of the real point set of a non-singular real algebraic curve of a given degree.

Gradually, this development made specialists change the traditional viewpoint. The problem of topological classification of placement $\mathbb{R}A \subset \mathbb{R}P^2$ for non-singular real algebraic curves A of a given degree, as the main problem of the topology of real plane algebraic curves, has been replaced by a finer problem of topological classification of placements $\mathbb{R}A \subset \mathbb{R}P^2$ taking into consideration not only degree of A, but also placement of $\mathbb{R}A$ in $\mathbb{C}A$.

Non-singular projective hypersurfaces can be considered as the most straightforward generalization of non-singular plane projective curves, and usually results on topology of non-singular real plane projective algebraic curves have more or less straightforward generalizations to the case of non-singular real projective hypersurfaces. Surfaces of the three-dimensional projective space are hypersurfaces next to plane curves.

Thus, it was natural to expect that the types of curves and the complex orientations of dividing curves are generalized to high-dimensional non-singular algebraic varieties, and, first of all, to non-singular real algebraic surfaces.

This problem was suggested to me by Rokhlin in late seventies, and I managed to find an answer. However, I could not generalize the most impressive applications of complex orientations of curves. Therefore I delayed a detailed publications, restricting myself to short announcements [19], [20], [21].

Now the subject attracts new people, see [6], [3]. In this volume several papers are devoted to or motivated by it. The original definitions and constructions almost are not mentioned there. Almost the same constructions look much more sophisticated. This is a natural process, but I do not like that the original faces of the subject have not appeared in literature. Therefore I decided to present the original approach with more or less complete motivations.

Contents

2 Generalization of type of a curve

2.1 Homological reformulation of the definition of type

How to reformulate the condition that

> the real point set $\mathbb{R}A$ of a non-singular curve A divides the complex point set $\mathbb{C}A$ of A

in such a way that this condition would make sense in high-dimensional case? The codimension of $\mathbb{R}A$ in $\mathbb{C}A$ is equal to the dimension of A. Thus if the dimension of A is greater than 1, and $\mathbb{R}A$ cannot divide $\mathbb{C}A$.

In the 1-dimensional case $\mathbb{R}A$ divides $\mathbb{C}A$ if and only if $\mathbb{R}A$ is zero-homologous in $\mathbb{C}A$, i. e. it realizes the trivial element of the group $H_1(\mathbb{C}A;\, \mathbb{Z}_2)$. This suggests the following definition:

A non-singular n-dimensional real algebraic variety A is said to *bound in complexification* if the set $\mathbb{R}A$ of its real points is zero-homologous in the set $\mathbb{C}A$ of complex points of A, i.e., realizes the trivial element of the group $H_n(\mathbb{C}A;\, \mathbb{Z}_2)$.

2.2 Relation to the form of the complex conjugation involution

Recall that if τ is an involution acting on an orientable manifold X of even dimension $2n$, then by the *form of* τ one calls the bilinear form

$$H_n(X) \times H_n(X) \to \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ \tau_* \beta$$

where $\circ$ denotes intersection number. This form is symmetric, if either n is even and τ preserves orientation of X, or n is odd and τ reverses orientation of X. Otherwise it is skew-symmetric. The inclusion $H_n(X) \otimes \mathbb{Z}_2 \to H_n(X;\, \mathbb{Z}_2)$ induces isometric embedding of its reduction modulo 2 into the similar $\mathbb{Z}_2$-*form of* τ defined by

$$H_n(X;\, \mathbb{Z}_2) \times H_n(X;\, \mathbb{Z}_2) \to \mathbb{Z}_2 : (\alpha, \beta) \mapsto \alpha \circ \tau_* \beta$$

2.2.A. Lemma (cf. Arnold [1]). *Under the conditions above, if the dimension of each component of the fixed point set F of τ is at most n, then the union of n-dimensional components of F realizes the characteristic class of the $\mathbb{Z}_2$-form of τ.*

(Remind that the characteristic class of a symmetric bilinear form $b : V\times V \to \mathbb{Z}_2$ on $\mathbb{Z}_2$-vector space V is a vector $\chi\in V$ such that $b(\xi,\xi)=b(\chi,\xi)$ for any $\xi\in V$. Any non-degenerate symmetric bilinear form has a unique characteristic class. This class is zero, if and only if the form is even (which means that $b(\xi,\xi)=0$ for any ξ).

Sketch of proof of 2.2.A. Take any class $\xi\in H_n(X;\mathbb{Z}_2)$, realize it by a cycle C transversal to F. (Here we assume, for simplicity, that X and τ are smooth or piecewise linear. Lemma is true in general situation, but we need it only for algebraic varieties.) By a small isotopy of C we can put C into general position with respect to $\operatorname{conj}(C)$. The intersection $C\cap\operatorname{conj}(C)$ is a finite set invariant with respect to conj. It consists of one-point orbits laying in F and two-point orbits disjoint with F. Therefore the number of points of $C\cap\operatorname{conj}(C)$ is congruent modulo 2 to the number of points of $C\cap F$. The first of these numbers reduced modulo 2 is $\xi\circ\operatorname{conj}_*\xi$, while the second one reduced modulo 2 is $\xi\circ\operatorname{conj}_*[F]$ □

Let A be a non-singular n-dimensional real algebraic variety, $\mathbb{R}A$ the set of its real points and $\mathbb{C}A$ the set of its complex points. Denote by conj the complex conjugation involution $\mathbb{C}A\to\mathbb{C}A$. The set $\mathbb{R}A$ is the fixed point set of conj.

According to Lemma 2.2.A, the homology class realized by $\mathbb{R}A$ is the characteristic class of the $\mathbb{Z}_2$-form of conj. It gives the following theorem.

2.2.B. Theorem. *Real algebraic variety A of even dimension bounds in complexification if and only if the $\mathbb{Z}_2$-form of complex conjugation involution* $\operatorname{conj}:\mathbb{C}A\to\mathbb{C}A$ *is even.* □

2.2.C. Corollary. *A real algebraic surface A of the 3-dimensional projective space having odd degree can not bound in complexification.*

Proof. Take any real plane section B of A. The complex point set $\mathbb{C}B$ of B is invariant under conj, and conj reverses orientation of $\mathbb{C}B$. Therefore the class $\beta\in H_2(\mathbb{C}A;\mathbb{Z}_2)$ realized by $\mathbb{C}B$ is invariant under conj_*. On the other hand it has self-intersection number $\beta\circ\beta$ equal to the degree of A (which is odd by assumption) reduced modulo 2. Therefore, the form of involution conj takes non-zero value $\beta\circ\operatorname{conj}_*(\beta)$ on it. □

These arguments admit straightforward generalization to the situation of a projective even-dimensional non-singular variety. Remind that the complex point set $\mathbb{C}A$ of projective variety A of dimension n realizes a non-zero class belonging to $H_n(\mathbb{C}P^N) = \mathbb{Z}$, this class is equal to d-fold generator of $H_n(\mathbb{C}P^N)$ (the latter is realized by $\mathbb{C}P^n$). The number d is called the *order* of A. It is equal to the intersection number of $\mathbb{C}A$ and $\mathbb{C}P^{N-n}$ in $\mathbb{C}P^N$. The order of a hypersurface is its degree, the order of a regular complete intersection of a collection of hypersurfaces equals the product of the degrees of these hypersurfaces.

2.2.D. Theorem (Generalization of 2.2.C). *A real projective even-dimensional non-singular variety of odd order can not bound in complexification.*

Proof. Let A be a real non-singular n-dimensional subvariety of order d of the projective N-dimensional space. Take a projective real $(N - \frac{n}{2})$-dimensional subspace P of the ambient space transversal to A. Denote the intersection of P with A by B. Since the self-intersection of B in A can be obtained as the intersection of A with the self-intersection of P in the ambient space, the self-intersection number of $\mathbb{C}B$ in $\mathbb{C}A$ is equal to d. Since $\mathbb{C}B$ is invariant under conj, the class $\beta \in H_{n/2}(\mathbb{C}A; \mathbb{Z}_2)$ is invariant under conj_*. It has non-zero self-intersection number, since it equals $d \mod 2$. Therefore the form of involution $\text{conj} : \mathbb{C}A \to \mathbb{C}A$ is not even. $\square$

After these assertions, which mean that under some conditions a real variety can not bound in complexification, it is natural to wonder, if it can bound in complexification in any case except the case of curves. Here is a sufficient condition, providing examples of varieties of any even dimension which bound in complexification.

2.2.E. Theorem. *Any even-dimensional non-singular M-variety with even intersection form of complexification bounds in complexification.*

Remind that a real non-singular algebraic variety A is called *M-variety,* if $\dim H_*(\mathbb{R}A; \mathbb{Z}_2) = \dim H_*(\mathbb{C}A; \mathbb{Z}_2)$. For any real algebraic variety A one has

$$\dim H_*(\mathbb{R}A; \mathbb{Z}_2) \leq \dim H_*(\mathbb{C}A; \mathbb{Z}_2). \qquad (1)$$

This inequality was found by R. Thom [17], he observed that it follows from a general Smith's theorem on homology of fixed point set of involution (see e. g. [2], Ch. III). It is a generalization of Harnack's inequality, which says that the number of components of a real point set $\mathbb{R}A$ of non-singular real algebraic curve A is at most $g(A) + 1$, where $g(A)$ is the genus of A. Inequality (1) is called *generalized Harnack inequality.* M-varieties are extremal cases of (1).

In [18] I constructed M-surfaces of any degree in 3-dimensional projective space.

2.2.F. Corollary. *Any non-singular M-surface of even degree in projective 3-space bounds in complexification.*

Proof. It follows from 2.2.E, since as it is well known, complexification of any surface A of even degree in 3-dimensional projective space has even intersection form. $\square$ $\square$

2.2.F shows that at least for any even $m > 0$ there exists a real non-singular surface in 3-dimensional projective space of degree m, which bounds in complexification.

Proof of 2.2.E. It is known (see Rokhlin [14]) that the complex conjugation involution of any M-variety A acts trivially in $H_*(\mathbb{C}A; \mathbb{Z}_2)$. Therefore the $\mathbb{Z}_2$-form of this involution coincides with the intersection form of $\mathbb{C}A$. $\square$

2.3 No more relation among homology classes of real components

The assertion that a non-singular real algebraic variety A bounds in classification can be reformulated by saying that the sum of $\mathbb{Z}_2$-homology classes realized by components of $\mathbb{R}A$ is equal to zero. It suggests a question, if it can happen that there are other relations among those classes. In the case of curves the answer is known to be negative.

Consider the case of A with $H_1(\mathbb{C}A; \mathbb{Z}_2) = 0$. The latter equality is not very restrictive assumption, since all projective regular complete intersections and cyclic branched covering of projective plane branched over non-singular plane curve satisfy it.

2.3.A. Theorem. *If A is a non-singular real algebraic surface A with* $H_1(\mathbb{C}A; \mathbb{Z}_2) = 0$*, then the kernel of the inclusion homomorphism* $H_2(\mathbb{R}A; \mathbb{Z}_2) \to H_2(\mathbb{C}A; \mathbb{Z}_2)$ *has dimension at most 1.*

Proof. Consider the Smith sequence of involution $\mathrm{conj} : \mathbb{C}A \to \mathbb{C}A$.

$$\begin{array}{ccccccc}
0 \longrightarrow & H_4(\mathbb{C}A/\operatorname{conj}, \mathbb{R}A) & \longrightarrow & H_4(\mathbb{C}A) & \longrightarrow & H_4(\mathbb{C}A/\operatorname{conj}, \mathbb{R}A) & \longrightarrow \\
& \| & & \| & & \| & \\
& \mathbb{Z}_2 & & \mathbb{Z}_2 & & \mathbb{Z}_2 &
\end{array}$$

$$\begin{array}{ccccccc}
\longrightarrow & H_3(\mathbb{C}A/\operatorname{conj},\mathbb{R}A) & \longrightarrow & H_3(\mathbb{C}A) & \longrightarrow & H_3(\mathbb{C}A/\operatorname{conj},\mathbb{R}A) & \longrightarrow \\
& \Vert & & \Vert & & \Vert & \\
& ? & & 0 & & \mathbb{Z}_2 &
\end{array}$$

$$\longrightarrow H_2(\mathbb{C}A/\operatorname{conj},\mathbb{R}A)\oplus H_2(\mathbb{R}A) \longrightarrow H_2(\mathbb{C}A)$$

The group $H_4(\mathbb{C}A)$ is $\mathbb{Z}_2$ since $\mathbb{C}A$ is a closed connected 4-manifold. The group $H_4(\mathbb{C}A/\operatorname{conj},\mathbb{R}A)$ is $\mathbb{Z}_2$ by the same reason (since $\mathbb{C}A/\operatorname{conj}$ is a closed connected manifold). From that, assumption $H_1(\mathbb{C}A;\,\mathbb{Z}_2)=0$ (which means by Poincaré duality that $H_3(\mathbb{C}A;\,\mathbb{Z}_2)=0$) and exactness of the Smith sequence it follows that the first boundary homomorphism of it

$$H_4(\mathbb{C}A/\operatorname{conj},\mathbb{R}A)\to H_3(\mathbb{C}A/\operatorname{conj},\mathbb{R}A)$$

is bijective, and therefore $H_3(\mathbb{C}A/\operatorname{conj},\mathbb{R}A)=\mathbb{Z}_2$. Consequently, the last homomorphism of the piece of the Smith sequence reproduced above has one-dimensional kernel. This homomorphism

$$H_2(\mathbb{C}A/\operatorname{conj},\mathbb{R}A)\oplus H_2(\mathbb{R}A)\to H_2(\mathbb{C}A)$$

restricted to the second summand is the inclusion homomorphism. Therefore the kernel of this inclusion homomorphism has dimension at most 1 (and it happens exactly when the image of the preceding homomorphism $H_3(\mathbb{C}A/\operatorname{conj},\mathbb{R}A)\to H_2(\mathbb{C}A/\operatorname{conj},\mathbb{R}A)\oplus H_2(\mathbb{R}A)$ of the Smith sequence is contained in the second summand $H_2(\mathbb{R}A)$). $\square$

3 Complex orientations of a real surface bounding in complexification

3.1 Digression: Arnold's complexification principle

I have to motivate my further considerations. Otherwise they look more tricky than they were. Unexpected help came from Arnold's speculations on complexifications.

Arnold told that everything in mathematics has a complexification. Some objects have obvious complexification. For example, the complexification of $\mathbb{R}$ is of course $\mathbb{C}$. Some complexifications are less obvious. The complexification of the group $\mathbb{Z}_2$ is $\mathbb{Z}$.

Sometimes complexification is not unique. For example, $S^1 = \mathbb{R}/\mathbb{Z}$ is a complexification of $\mathbb{Z}_2$ too. Indeed, S^1 is isomorphic to the multiplicative group of complex numbers with absolute value 1, while $\mathbb{Z}_2$ is isomorphic to the multiplicative group of real numbers with absolute value 1. This non-uniqueness of complexification is explained by the fact that $\mathbb{Z}$ and S^1 are dual to each other (each of them is the character group for the other), while the group $\mathbb{Z}_2$ is self-dual.

Complexification of symmetric group S_n is the braid group B_n. (Note that this agrees with the previous example: $S_2 = \mathbb{Z}_2$ and $B_2 = \mathbb{Z}$!)

Even the most "real" notions have complexifications. Consider the notion of *inequality.* What is its complexification? To answer to this question, let us reformulate the statement that $a \geq 0$ without the sign $\geq$. It is easy: assertion $f \geq 0$ is equivalent to the assertion that there exists a real ξ such that $f = \xi^2$. Inequality is related to the notion of *manifold with boundary* (generic inequality defines a manifold with boundary). In the complex domain equation $f = \xi^2$ is related to the notion of *double branched covering*: if the equation $f(x_1, \dots, x_n) = 0$ determines (locally or globally) a complex hypersurface A of a complex manifold B (where $x_1, \dots, x_n$ are coordinates), then the equation $f(x_1, \dots, x_n) = \xi^2$ defines in an ambient space with coordinates $x_1, \dots, x_n$, ξ a manifold X, which two-fold covers B with projection defined by $(x_1, \dots, x_n, \xi) \mapsto (x_1, \dots, x_n)$.

Thus the complexification of the notion of manifold with boundary is the notion of two-fold branched covering. Here the boundary is complexified by the branch locus.

In some situations this transition from a manifold with boundary to a double branched covering has a concrete form. For example, let A be a real non-singular plane projective curve of an even degree $2k$ defined by an equation $f(x_0, x_1, x_2) = 0$. Then the equation $f(x_0, x_1, x_2) = \xi^2$ defines a non-singular algebraic surface in a weighted quasi-projective space. The set of its complex points is a two-fold covering space of $\mathbb{C}P^2$ with branching locus $\mathbb{C}A$. The set of its real points is projected to a part of $\mathbb{R}P^2$ defined by inequality $f(x_0, x_1, x_2) \geq 0$.

It was this construction that was used by Arnold in his breakthrough work [1] to get relations between topology of pair $(\mathbb{R}P^2, \mathbb{R}A)$ and topology of 4-dimensional manifold $\mathbb{C}A$ with involution conj.

3.2 Complexification of the notion of boundary does work, suggesting definition of complex orientations of surface

Transition from real curves to real surfaces resembles slightly complexification. At least all dimensions are multiplied by 2.

In the definition of complex orientations of a real curve the crucial step was to consider the set of real points of the curve as the boundary of a half of the set of its complex points. A happy idea was to look for "complexification" of this construction.

Let A be a non-singular real algebraic surface bounding in complexification. By definition it means that $\mathbb{R}A$ realizes $0 \in H_2(\mathbb{C}A; \mathbb{Z}_2)$. According to Arnold's philosophy, one should find a double covering of $\mathbb{C}A$ branched over $\mathbb{R}A$.

3.2.A. Theorem. *There exists a double covering of $\mathbb{C}A$ branched over $\mathbb{R}A$. This covering is unique up to equivalence, if $H_1(\mathbb{C}A; \mathbb{Z}_2) = 0$.*

This assertion follows from the following well-known classification theorem.

3.2.B. Theorem. *Let A be a closed n-submanifold of an $(n+2)$-manifold X. Double coverings of X branched over A considered up to equivalence are in one to one correspondence with homology classes $\eta \in H_{n+1}(X, A; \mathbb{Z}_2)$ such that $\partial(\eta) = [A] \in H_n(A; \mathbb{Z}_2)$.*

I will not prove here 3.2.B, but remind the most classic way to construct such a covering. Cut the ambient manifold X along a chain C realizing η. Take two copies of the result, and glue them identifying copies of opposite edges of the cut. The result is the desired covering space.

Deduce 3.2.A from 3.2.B. Consider homology sequence of $(\mathbb{C}A, \mathbb{R}A)$:

$$H_3(\mathbb{C}A; \mathbb{Z}_2) \to H_3(\mathbb{C}A, \mathbb{R}A; \mathbb{Z}_2) \xrightarrow{\partial} H_2(\mathbb{R}A; \mathbb{Z}_2) \xrightarrow{in_*} H_2(\mathbb{C}A; \mathbb{Z}_2).$$

By 2.3.A the kernel of in_* contains at most one non-zero element, and since A bounds in complexification, $\ker(in_*)$ contains $[\mathbb{R}A]$. By exactness of the homology sequence above, there exists $\eta \in H_3(\mathbb{C}A, \mathbb{R}A; \mathbb{Z}_2)$ with

$\partial(\eta) = [\mathbb{R}A] \in H_2(\mathbb{R}A; \mathbb{Z}_2)$. If $H_1(\mathbb{C}A; \mathbb{Z}_2) = 0$, then by Poincaré duality $H_3(\mathbb{C}A; \mathbb{Z}_2) = 0$, and such η is unique. $\square$

For the sake of simplicity consider the case $H_1(\mathbb{C}A; \mathbb{Z}_2) = 0$. Denote the double covering of $\mathbb{C}A$ branched over $\mathbb{R}A$ by $DA \to \mathbb{C}A$. Involution $\operatorname{conj} : \mathbb{C}A \to \mathbb{C}A$ preserves $\mathbb{R}A$ and therefore the induced involution preserves class $\eta \in H_3(\mathbb{C}A, \mathbb{R}A; \mathbb{Z}_2)$ with $\partial(\eta) = [\mathbb{R}A] \in H_2(\mathbb{R}A; \mathbb{Z}_2)$ characterizing the covering. Consequently conj can be lifted to DA. In fact there are two liftings of it differing from each other by the non-trivial automorphism of the covering space DA. Denote them by c_+ and c_-.

Consider a fiber D of a tubular neighborhood of $\mathbb{R}A$ in $\mathbb{C}A$. It is homeomorphic to D^2 and without loss of generality one can assume that $\operatorname{conj}(D) = D$ and conj acts in D as symmetry with respect to the center $D \cap \mathbb{R}A$ of D. The preimage $\tilde{D}$ of D in DA is a fiber of a tubular neighborhood of $\mathbb{R}A$ in DA. The liftings c_+ and c_- of conj act in $\tilde{D}$ as rotations by $\pm\pi/2$, since they cover the symmetry. Each of them determines an orientation of $\tilde{D}$, namely the orientation with respect to which this is rotation by $\pi/2$ in positive direction. Thus c_+ and c_- determine two orientations of the normal bundle of $\mathbb{R}A$ in DA. These orientations are opposite to each other. Manifold DA is naturally oriented: there is orientation with respect to which the projection $DA \to \mathbb{C}A$ has degree $+2$. Therefore the orientations of normal bundle of $\mathbb{R}A$ in DA above determine orientations of $\mathbb{R}A$. This pair of opposite orientations is the desired pair of complex orientations of A.

If $\mathbb{R}A$ is not empty, then the liftings c_+ and c_- of conj are transformations of order 4. Indeed, their squares c_+^2, c_-^2 cover the transformation $\operatorname{conj}^2 = \mathrm{id}$ and are non-trivial, since they act non-trivially in $\tilde{D}$. Therefore they are non-trivial automorphisms of the covering $DA \to \mathbb{C}A$ and their squares are identity.

The orbit spaces of them coincide with $\mathbb{C}A/\operatorname{conj}$ and the projection $DA \to \mathbb{C}A/\operatorname{conj}$ is the composition of $DA \to \mathbb{C}A \to \mathbb{C}A/\operatorname{conj}$. It is a cyclic 4-fold covering branched over $\mathbb{R}A$.

This covering gives another definition of the complex orientations of A. Namely, the 4-fold cyclic covering of $\mathbb{C}A/\operatorname{conj}$ branched over A gives an element of $H^1(\mathbb{C}A/\operatorname{conj} \smallsetminus \mathbb{R}A; \mathbb{Z}_4) = \hom(H_1(\mathbb{C}A/\operatorname{conj} \smallsetminus \mathbb{R}A), \mathbb{Z}_4)$, which is a characteristic class of that covering. The dual homology class belongs to $H_3(\mathbb{C}A/\operatorname{conj}, \mathbb{R}A; \mathbb{Z}_4)$. Its image under the boundary homomorphism

$$H_3(\mathbb{C}A/\operatorname{conj}, \mathbb{R}A; \mathbb{Z}_4) \to H_2(\mathbb{R}A; \mathbb{Z}_4)$$

is a fundamental class of $\mathbb{R}A$, since the branch index at each point of $\mathbb{R}A$ is

4. This class is lifted to an orientation class belonging to $H_2(\mathbb{R}A)$. It is not difficult to show that it is one of the complex orientations of A defined above. This construction gives both complex orientations, since the characteristic class of the 4-fold covering is defined up to sign.

3.3 Kharlamov's congruence

Existence of the complex orientations defined in the preceding section provides immediately a simple proof of the following theorem, which was obtained first by Kharlamov [7].

3.3.A. Theorem. *Kharlamov congruence If A is a non-singular real algebraic surface which bounds in its complexification and has $H_1(\mathbb{C}A; \mathbb{Z}_2) = 0$, then*

$$\chi(\mathbb{R}A) \equiv 0 \mod 8$$

Proof. The class realized by $\mathbb{R}A$ in $\mathbb{C}A/\operatorname{conj}$ is divisible by 4, because there exists the cyclic 4-fold covering of $\mathbb{C}A/\operatorname{conj}$ branched over $\mathbb{R}A$. Therefore the self-intersection number of $\mathbb{R}A$ in $\mathbb{C}A/\operatorname{conj}$ is divisible by 16. On the other hand this self-intersection number is equal to the self-intersection number of $\mathbb{R}A$ in $\mathbb{C}A$ multiplied by 2, and the self-intersection number of $\mathbb{R}A$ in $\mathbb{C}A$ is $-\chi(\mathbb{R}A)$. $\square$ $\square$

3.4 Complex orientations and classes lifted from the orbit space of complex conjugation

Let A be a non-singular real algebraic surface with $H_1(\mathbb{C}A; \mathbb{Z}_2) = 0$. Consider the inverse Hopf homomorphism

$$p^! : H_2(\mathbb{C}A/\operatorname{conj}) \to H_2(\mathbb{C}A)$$

induced by the natural projection $p : \mathbb{C}A \to \mathbb{C}A/\operatorname{conj}$. (Remind that it can be defined as the composition

$$\begin{tikzcd}
H_2(\mathbb{C}A/\operatorname{conj}) \arrow[r, "\text{Poincaré duality}"] & H^2(\mathbb{C}A/\operatorname{conj}) \arrow[d, "p^*"] \\
H_2(\mathbb{C}A) & H^2(\mathbb{C}A) \arrow[l, "\text{Poincaré duality}"']
\end{tikzcd}$$

and geometrically be described as assignment to the class of a surface transversal to $\mathbb{R}A$ the class of its preimage under p.)

The composition $p_* \circ p^!$ is the multiplication by the degree of p, i. e. by 2. Since $H_1(\mathbb{C}A; \mathbb{Z}_2) = 0$, from the universal coefficient formula it follows that $H_2(\mathbb{C}A)$ has no elements of order 2. Therefore $p^! : H_2(\mathbb{C}A/\operatorname{conj}) \to H_2(\mathbb{C}A)$ is a monomorphism. Its image consists of classes invariant under conj_*, but some classes invariant under conj_* do not belong to the image. Moreover, in terms of this image it is possible to give the following description of the complex orientations.

3.4.A. Theorem. *Let A be a non-singular real algebraic surface with $H_1(\mathbb{C}A; \mathbb{Z}_2) = 0$ which bounds in complexification. The complex orientations of A are the only orientations such that the class $\alpha \in H_2(\mathbb{C}A)$ realized by $\mathbb{R}A$ equipped with the orientation is equal to 2β with $\beta \in p^!(H_2(\mathbb{C}A/\operatorname{conj}))$.*

3.4.B. Lemma. *Multiplication by 2 transforms any class $\alpha \in H_2(\mathbb{C}A)$ which is invariant under* conj_* *into a class belonging to the image of $p^!$.*

Proof of 3.4.B. Note, first, that the situation in homology with coefficients in $\mathbb{Z}[1/2]$ is simpler. The image of the inverse Hopf homomorphism

$$p^!_{\mathbb{Z}[1/2]} : H_2(\mathbb{C}A/\operatorname{conj}; \mathbb{Z}[1/2]) \to H_2(\mathbb{C}A; \mathbb{Z}[1/2])$$

coincides with the set of classes invariant under conj_*. See [2].

On the other hand, since $H_1(\mathbb{C}A; \mathbb{Z}_2) = 0$, there is no 2-torsion in $H_2(\mathbb{C}A)$ and therefore the coefficient homomorphism $H_2(\mathbb{C}A) \to H_2(\mathbb{C}A; \mathbb{Z}[1/2])$ induced by the inclusion $\mathbb{Z} \to \mathbb{Z}[1/2]$ is a monomorphism.

Since $\mathbb{R}A \neq \varnothing$, projection $\mathbb{C}A \to \mathbb{C}A/\operatorname{conj}$ induces epimorphism

$$H_1(\mathbb{C}A; \mathbb{Z}_2) \to H_1(\mathbb{C}A/conj; \mathbb{Z}_2)$$

and therefore $H_1(\mathbb{C}A/\operatorname{conj}; \mathbb{Z}_2) = 0$. Thus for the same reason as above, the coefficient homomorphism $H_2(\mathbb{C}A/\operatorname{conj}) \to H_2(\mathbb{C}A/\operatorname{conj}; \mathbb{Z}[1/2])$ is injective. The coefficient homomorphisms commute with p_* and $p^!$ Therefore in what follows we may identify elements of $H_2(\mathbb{C}A)$ and $H_2(\mathbb{C}A/\operatorname{conj})$ with their images in $H_2(\mathbb{C}A; \mathbb{Z}[1/2])$ and $H_2(\mathbb{C}A/\operatorname{conj}; \mathbb{Z}[1/2])$ under the coefficient homomorphisms.

Thus α is an image of some class $\beta \in H_2(\mathbb{C}A/\operatorname{conj}; \mathbb{Z}[1/2])$, and therefore $2\alpha = p^!(2\beta)$. However $2\beta = p_* \circ p^!(\beta) = p_*(\alpha) \in p_* H_2(\mathbb{C}A) \subset H_2(\mathbb{C}A/\operatorname{conj})$, and $2\alpha \in p^! H_2(\mathbb{C}A/\operatorname{conj}) \subset H_2(\mathbb{C}A)$. $\square$ $\square$

Proof of 3.4.A. By the definition of complex orientations, the real part $\mathbb{R}A$ equipped with a complex orientation realizes $0 \in H_2(\mathbb{C}A/\operatorname{conj}; \mathbb{Z}_4)$. Therefore in $H_2(\mathbb{C}A/\operatorname{conj})$ this surface with the complex orientation realizes class divisible by 4. Denote this class by γ and the result of the division by δ, so $\gamma = 4\delta$. The class $2\alpha \in H_2(\mathbb{C}A)$ is equal to $p^!(\gamma)$. Therefore $\alpha = 2p^!(\delta)$.

Suppose now that a class $\alpha' \in H_2(\mathbb{C}A)$, which is realized by $\mathbb{R}A$ equipped with some orientation, is equal to $2p^!(\delta')$. Then $2\alpha'$ is an image of the class $\gamma' \in H_2(\mathbb{C}A/\operatorname{conj})$ realized by $\mathbb{R}A$ with the same orientation. Since $p^!$ is injective, $\gamma' = 4\delta'$. Divisibility of γ' by 4 gives existence of 4-fold covering of $\mathbb{C}A/\operatorname{conj}$ branched over $\mathbb{R}A$. It is easy to see that one of generators of its automorpism group defines on $\mathbb{R}A$ the orientation which gives α'. But as it was shown above, such a covering is unique and the orientation should be one of the complex ones. □ □

The material of this section emerged in a talk with Kharlamov in the beginning of July 1980.

3.5 Remark on homology description of a real algebraic surface

In papers on topology of real algebraic K3 surfaces (see [8], [9], [13], [10]) the topology is characterized usually by the following homological data: integer homology $H_2(\mathbb{C}A)$ of the complexification, intersection form $H_2(\mathbb{C}A) \times H_2(\mathbb{C}A) \to \mathbb{Z}$, and its isometry $\operatorname{conj}_* : H_2(\mathbb{C}A) \to H_2(\mathbb{C}A)$ induced by conj. Its curious that these data in the case of K3 surfaces are really sufficient for description even up to rough projective equivalence (i. e. rigid isotopy and projective isomorphism). Thus the image of $p^!$ does not contain any new information in the case, and the complex orientations can be restored from the homology data above. This makes the following questions interesting:

(1) Is that a special property of K3 surfaces, or it can be generalized to some wider class of real algebraic surfaces?

(2) How are the complex orientations of K3 surfaces restored from the homology data?

4 Relative complex orientations of a real surface

4.1 Types of real algebraic surfaces revised

In the case of curve the group $H_1(\mathbb{C}A;\mathbb{Z}_2)$ contains only one naturally distinguished element: zero. Consequently, there are only two types of real algebraic curves: curves of type I having $[\mathbb{R}A] = 0 \in H_1(\mathbb{C}A;\mathbb{Z}_2)$ and curves of type II having $[\mathbb{R}A] \neq 0 \in H_1(\mathbb{C}A;\mathbb{Z}_2)$.

In the case of surface, the group $H_2(\mathbb{C}A;\mathbb{Z}_2)$ contains at least one other naturally distinguished element: the class of a hyperplane section. As it is well known, it is not zero. Thus a new type of real algebraic surfaces appears.

A real algebraic surface A is said to be of *type* I_{rel}*,* if the $\mathbb{R}A$ is homologous mod 2 to a hyperplane section. A real algebraic surface, which bounds in complexification, is said to be of *type* I_{abs}. All other real algebraic surfaces are said to be of *type II.*

For some surfaces one can find other remarkable elements of $H_2(\mathbb{C}A;\mathbb{Z}_2)$. Namely, there may be classes realized by (complex) algebraic curves distinct from the class of hyperplane sections. Phenomena related to the fact that the real part of a surface can be homologous to complex cycles realized by algebraic curves deserves special investigation. However a generic surface of general type has no classes realized by algebraic cycles distinct from the class of hyperplane section, and therefore I do not feel necessity to introduce a collection of types finer than one given above.

4.1.A. Lemma. *Real algebraic surface A is of type I_{rel} if and only if the class of hyperplane section is the characteristic class of the $\mathbb{Z}_2$-form of the involution* $\operatorname{conj}:\mathbb{C}A\to\mathbb{C}A$*.*

Proof. It follows from Lemma 2.2.A and uniqueness of the characteristic class. □ □

4.1.B. Theorem. *Any non-singular M-surface of odd degree in 3-dimensional projective space is of type I_{rel}.*

Proof. It follows from 4.2.A since, as it is well known, the class of plane section of a surface of odd degree in 3-dimensional projective space is the characteristic class of the $\mathbb{Z}_2$-intersection form, and for an M-surface the $\mathbb{Z}_2$-intersection form coincides with the $\mathbb{Z}_2$-form of complex conjugation involution. □ □

4.2 Complex orientations of a real surface modulo a curve

Now it is natural to expect some analog of complex orientations for surfaces of type I_{rel}. However instead of an analog we find a generalization.

Let A be a non-singular real algebraic surface and C a real algebraic curve on A such that $\mathbb{C}C$ and $\mathbb{R}A$ realize the same $\mathbb{Z}_2$-homology class. As above, assume that $H_3(\mathbb{C}A;\mathbb{Z}_2) = 0$. (The situation considered in Section 3.2 appears a special case of this one, if one allows the curve C to be empty.)

Under these assumptions, the construction described below gives two orientations of $\mathbb{R}A \setminus \mathbb{R}C$ opposite to each other. They are called *complex orientations of A modulo C*.

4.2.A. Lemma. *There exists a unique two-fold covering of $\mathbb{C}A$ branched over $\mathbb{R}A \cup \mathbb{C}C$.*

Here the total space of the covering is not a manifold. It has singularities over the singularities of the branch locus, i. e. $\mathbb{R}C = \mathbb{R}A \cap \mathbb{C}C$. By covering branched over $\mathbb{R}A \cup \mathbb{C}C$ we mean a natural extension of a covering over the complement $\mathbb{C}A \setminus (\mathbb{R}A \cup \mathbb{C}C)$ of the branched locus such that the covering can not be extended to a covering of $(\mathbb{C}A \setminus (\mathbb{R}A \cup \mathbb{C}C)) \cup pt$ for any point $pt \in \mathbb{R}A \cup \mathbb{C}C$.

Proof of 4.2.A. We can use a slight generalization of 3.2.B to the case when the branch locus is a union of two submanifolds. According to that generalization, coverings under consideration are in one to one correspondence with homology classes $\eta \in H_3(\mathbb{C}A, \mathbb{R}A \cup \mathbb{C}C; \mathbb{Z}_2)$ with $\partial\eta = [\mathbb{R}A] + [\mathbb{C}C]$. Consider the following segment of the homology sequence of pair $(\mathbb{C}A, \mathbb{R}A \cup \mathbb{C}C)$:

$$H_3(\mathbb{C}A;\mathbb{Z}_2) \to H_3(\mathbb{C}A, \mathbb{R}A \cup \mathbb{C}C;\mathbb{Z}_2) \to H_2(\mathbb{R}A \cup \mathbb{C}C;\mathbb{Z}_2) \to H_2(\mathbb{C}A;\mathbb{Z}_2).$$

Existence of η follows from the assumption that $[\mathbb{R}A] + [\mathbb{C}C]$ is mapped by the last homomorphism to zero. Uniqueness follows from the assumption that $H_3(\mathbb{C}A;\mathbb{Z}_2) = 0$. □ □

The rest of construction runs as in the absolute case. A reader, who feels uncomfortable with singular branched covering, may first delete $\mathbb{C}C$. Singularities would be deleted, and the situation would be the same as in Section 3.2, but all varieties become non-compact.

The first obvious question on complex orientations of a surface A modulo a curve C is how they are organized in a neighborhood of $\mathbb{R}C$. One can imagine two opportunities: an orientation of $\mathbb{R}A \smallsetminus \mathbb{R}C$ can be extendible or not extendible across $\mathbb{R}C$.

4.2.B. Theorem. *A complex orientation of a surface A modulo a curve C is not extendible to an orientation of $\mathbb{R}A$.*

Proof. First, note that locally all objects involved in the construction of the orientations above are standard. Indeed, consider a point $pt \in \mathbb{R}C$. Since it is non-singular for both A and C, it has a neighborhood U in $\mathbb{C}A$ such that there exists a diffeomorphism $h : U \to D^4$ with

1. $h \circ \operatorname{conj} \circ h^{-1} : (x_1, x_2, x_3, x_4) \mapsto (x_1, -x_2, x_3, -x_4)$,
2. $h(U \cap \mathbb{C}C) = \{x \in D^4 \mid x_3 = x_4 = 0\}$,
3. $h(U \cap \mathbb{R}A) = \{x \in D^4 \mid x_2 = x_4 = 0\}$.

Therefore behavior of a complex orientation of A modulo C at pt should be standard too. Now one can trace the construction above in the model case, but we will use slightly easier indirect arguments: consider the example with A being projective plane and C a projective line. By 4.1.B, $\mathbb{R}P^2$ and $\mathbb{C}P^1$ are homologous modulo 2 in $\mathbb{C}P^2$, and thus projective plane has complex orientations modulo a projective line. Since $\mathbb{R}P^2$ is not orientable, a complex orientation of it modulo line can not be extended across $\mathbb{R}C$. □ □

4.2.C. Corollary. *If A is a non-singular real algebraic surface with trivial $H_1(\mathbb{C}A; \mathbb{Z}_2)$ and C is a non-singular curve on it such that $[\mathbb{R}A] + [\mathbb{C}C] = 0 \in H_2(\mathbb{C}A; \mathbb{Z}_2)$, then $\mathbb{R}C$ realizes the element of $H_1(\mathbb{R}A; \mathbb{Z}_2)$ dual to the first Stiefel-Whitney class of $\mathbb{R}A$.*

Proof. It follows from existence of complex orientation of A modulo C which is an orientation of $\mathbb{R}A \smallsetminus \mathbb{R}C$ and impossibility to extend it across $\mathbb{R}C$. □ □

4.2.D. Corollary. *The set of real points of any non-singular real projective surface of even degree and type I_{rel} in 3-dimensional projective space is contractible in $\mathbb{R}P^3$.*

Proof. Since $\mathbb{R}P^3$ is orientable, and the set of real points $\mathbb{R}A$ of any surface A of even degree in $\mathbb{R}P^3$ divides $\mathbb{R}P^3$ into two pieces, $\mathbb{R}A$ is orientable as the common boundary of these two halves of $\mathbb{R}P^3$. Thus any curve in $\mathbb{R}A$ realizing the element of $H_1(\mathbb{R}A; \mathbb{Z}_2)$ dual to the Stiefel-Whitney class of $\mathbb{R}A$ should bound in $\mathbb{R}A$.

Now consider any non-singular plane section C of A. By 4.2.C $\mathbb{R}C$ realizes element dual to the Stiefel-Whitney class. Thus $\mathbb{R}C$ should bound in $\mathbb{R}A$. Therefore the intersection number of $\mathbb{R}C$ and any loop on $\mathbb{R}A$ is zero. But this number is equal to the intersection number of the loop and plane in $\mathbb{R}P^3$. Consequently, any loop on $\mathbb{R}A$ is contractible in $\mathbb{R}P^3$. It follows that $\mathbb{R}A$ is contractible in $\mathbb{R}P^3$ itself. $\square$ $\square$

Remark. There exist surfaces of even degree and type I_{rel} contractible in $\mathbb{R}P^3$. The simplest surface of this kind is the usual sphere. In fact, for it the group $H_2(\mathbb{C}A; \mathbb{Z}_2)$ is generated by the classes of complex conjugate lines in it. A class of plane section can be realized by union of a line of one family and a line of the other family. Since lines of the same family are pairwise disjoint and each line of one family intersects each line of the other family in one point and transversally, the intersection number of the class of plane section and the base classes is one. On the other hand, exactly one line of one family and one line of the other family pass through each real point of the sphere. Therefore the intersection number of $[\mathbb{R}A]$ with both base classes is also one. Since any homology class is characterized by its intersection numbers with the base classes, $[\mathbb{R}A]$ equals the plane section class and sphere is of type I_{rel}.

4.3 Semi-orientations

Pairs of orientations opposite to each other occur so frequently here, that one feels a necessity to introduce a term. A pair of orientations opposite to each other will be called *semi-orientation.*

The structure contained in a semi-orientation is equivalent to a construction assigning to a local orientation at some point x_1 a local orientation of at any other point x_2. This assignment should satisfy the following conditions.

1. Reversion of the local orientation at x_1 implies reversion of the corresponding local orientation at x_2.

2. For any three points x_1, x_2 and x_3 the local orientation at x_3 obtained by the construction from a local orientation at x_1 directly coincides with the local orientation obtained from the same local orientation at

x_1 in two steps: first, constructing the corresponding local orientation at x_2 and then applying the construction to the latter local orientation at x_2.

3. If x_1 and x_2 are connected by path, then the construction coincides with the transfer of local orientation along the path.

Any construction satisfying those conditions gives rise to pair of orientations opposite to each other. Namely, local orientations obtained by the construction from one local orientation at some point x constitute an orientation, and starting from the other orientation at x one get the opposite orientation. It is easy to see, that the pair of orientations does not depend on the choice of x.

On the other hand, any pair of orientations opposite to each other gives rise to a construction satisfying the conditions above. Namely, the construction assigning to a local orientation at x_1 the local orientation at x_2 such that both local orientations agree with the same orientation of the given pair of orientations opposite to each other.

4.4 Internal definition of complex orientations (without 4-fold coverings)

Let A be a non-singular real algebraic surface with $H_1(\mathbb{C}A; \mathbb{Z}_2) = 0$ and C be a non-singular real algebraic curve in A with $[\mathbb{C}C] + [\mathbb{R}A] = 0 \in H_2(\mathbb{C}A; \mathbb{Z}_2)$. We admit the case of empty C.

According to Section 4.2, there is a complex semi-orientation of A modulo C opposite to each other. In this section we consider an alternative construction for it. It is described in terms of the corresponding assignment of local orientations (see Section 4.3 above).

Let x_1 and x_2 be two points of $\mathbb{R}A \smallsetminus \mathbb{R}C$. Denote by D_i the fiber of tubular neighborhood of $\mathbb{R}A$ over the point x_i and by S_i the boundary circle of D_i. Take a point y_i in S_i.

Since $\dim \mathbb{C}A = 4$ and $\dim(\mathbb{R}A \cup \mathbb{C}C) = 2$, the space $\mathbb{C}A \smallsetminus (\mathbb{R}A \cup \mathbb{C}C)$ is connected. Choose a path $s : I \to \mathbb{C}A \smallsetminus (\mathbb{R}A \cup \mathbb{C}C)$ connecting y_1 with y_2.

Consider now some local orientations of $\mathbb{R}A$ at x_1 and x_2. A local orientation of $\mathbb{R}A$ at x_i defines an orientation of D_i such that the local intersection number of $\mathbb{R}A$ and D_i at x_i is $+1$. This orientation of D_i defines an orientation of S_i (since $S_i = \partial D_i$). Let u_i be a path on S_i with $u_i(0) = y_i$ and $u_i(1) = \mathrm{conj}(y_i)$ which agrees with the orientation of S_i.

Consider the loop $su_2(\mathrm{conj}\circ s)^{-1}u_1^{-1}$. It is zero-homologous in $\mathbb{C}A$ modulo 2, since by hypothesis $H_1(\mathbb{C}A;\mathbb{Z}_2)=0$. Therefore the linking number of it with $\mathbb{R}A\cup\mathbb{C}C$ is well defined (modulo 2).

4.4.A. Lemma. *The linking number of the loop $su_2(\mathrm{conj}\circ s)^{-1}u_1^{-1}$ and $\mathbb{R}A\cup\mathbb{C}C$ is zero, iff the local orientations of $\mathbb{R}A$ involved in construction of u_i agree with the same complex orientation of A modulo C.*

Proof. Let $Y\to\mathbb{C}A$ be the two-fold covering branched over $\mathbb{R}A\cup\mathbb{C}C$. Choose a point $\tilde y_1\in Y$ over y_1 and construct a path $v:I\to Y$ with $v(0)=\tilde y_1$ covering s. Denote $v(1)$ by $\tilde y_2$. Obviously it lies over y_2. Construct paths $\tilde u_i$ covering u_i and starting at $\tilde y_i$. Denote the point $\tilde u_i(1)$ by z_i.

Assume that the local orientations agree with one of two complex orientation. Then z_i is the image of $\tilde y_i$ under the transformation $\widetilde{\mathrm{conj}}:Y\to Y$ covering $\mathrm{conj}:\mathbb{C}A\to\mathbb{C}A$, and path $\widetilde{\mathrm{conj}}\circ v$ joins points z_1 and z_2. Therefore path $v\tilde u_2(\widetilde{\mathrm{conj}}\circ v)^{-1}\tilde u_1^{-1}$ is a loop covering loop $su_2(\mathrm{conj}\circ s)^{-1}u_1^{-1}$. It means that the latter loop has zero linking number with $\mathbb{R}A\cup\mathbb{C}C$.

If the local orientations do not agree with each of the complex orientations, then reversing local orientation at x_1 make them agree with one of the complex orientations. This changes loop $su_2(\mathrm{conj}\circ s)^{-1}u_1^{-1}$ by the loop running once along S_2, and make linking number of $su_2(\mathrm{conj}\circ s)^{-1}u_1^{-1}$ with $\mathbb{R}A\cup\mathbb{C}C$ to be zero. Since linking number of S_2 and $\mathbb{R}A\cup\mathbb{C}C$ is 1, the original linking number of $su_2(\mathrm{conj}\circ s)^{-1}u_1^{-1}$ with $\mathbb{R}A\cup\mathbb{C}C$ was 1. $\square$ $\square$

To define complex semi-orientation of A modulo C in a manner of Section 4.3, I have to construct for any local orientation of $\mathbb{R}A$ at x_1 a local orientation of $\mathbb{R}A$ at x_2 in such a way that the construction satisfies the conditions of Section 4.3. Lemma 4.4.A suggests such a construction. For any local orientation of $\mathbb{R}A$ at x_1 one should choose a local orientation of $\mathbb{R}A$ at x_2 such that the loop provided by the construction above has zero linking number with $\mathbb{R}A\cup\mathbb{C}C$. It follows from Lemma 4.4.A that this construction gives the desired complex semi-orientation of A modulo C.

Note that the conditions of Section 4.3 for this construction can be easily verified independently of Lemma 4.4.A and the original construction of complex orientations. Thus the construction of this section can be used as a base for the whole theory.

4.5 Orientations modulo changing curve

In this section we study behavior of complex orientation of a surface modulo curve when the curve moves.

4.5.A. Theorem. *Let A be a non-singular real algebraic surface with $H_1(\mathbb{C}A;\mathbb{Z}_2)=0$ and C_1, C_2 be two non-singular real algebraic curves on A with $\mathbb{C}C_1$ and $\mathbb{C}C_2$ realizing the same element of $H_2(\mathbb{C}A;\mathbb{Z}_2)$ as $\mathbb{R}A$. Then $\mathbb{R}C_1\cup\mathbb{R}C_2$ divides $\mathbb{R}A$ into two parts (which are the images of sets of real points of the two-fold coverings of $\mathbb{C}A$ branched over $\mathbb{C}C_1\cup\mathbb{C}C_2$). Any complex orientation of A modulo C_1 and a complex orientation of A modulo C_2 coincide on one of these parts and are opposite on the other.*

Proof. Take a point $x_1\in\mathbb{R}A\setminus(\mathbb{R}C_1\cup\mathbb{R}C_2)$. Reversing, if necessary, the complex orientation of A modulo C_2, one may assume that the complex orientation of A modulo C_2 coincides at x_1 with the complex orientation of A modulo C_1. Take any point $x_2\in\mathbb{R}A\setminus(\mathbb{R}C_1\cup\mathbb{R}C_2)$. For local orientations at x_1 and x_2 induced by the complex orientation of A modulo C_1, apply the construction of Section 4.4 choosing a path s with $s(I)$ disjoint from $\mathbb{R}A\cup\mathbb{C}C_1\cup\mathbb{C}C_2$. By Lemma 4.4.A, the linking number of loop $su_2(\operatorname{conj}\circ s)^{-1}u_1^{-1}$ with $\mathbb{R}A\cup\mathbb{C}C_1$ is zero. The linking number of the same loop with $\mathbb{R}A\cup\mathbb{C}C_2$ is zero, iff at x_2 the complex orientation of A modulo C_2 coincides with the complex orientation of A modulo C_1. On the other hand, the linking number of the that loop with $\mathbb{R}A\cup\mathbb{C}C_2$ is equal to the linking number of it with $\mathbb{C}C_1\cup\mathbb{C}C_2$. The latter depends only on x_2.

To complete the proof we have to show that the dependence is as in 4.5.A, i. e. that the set of $x_2\in\mathbb{R}A\setminus(\mathbb{R}C_1\cup\mathbb{R}C_2)$ such that the linking number of loop $su_2(\operatorname{conj}\circ s)^{-1}u_1^{-1}$ with $\mathbb{C}C_1\cup\mathbb{C}C_2$ is zero coincides with the image of the set of real points of the two-fold covering of $\mathbb{C}A$ branched over $\mathbb{C}C_1\cup\mathbb{C}C_2$.

The covering does exit since $\mathbb{C}C_1$ and $\mathbb{C}C_2$ realize the same element of $H_2(\mathbb{C}A;\mathbb{Z}_2)$. It is unique, since $H_1(\mathbb{C}A;\mathbb{Z}_2)=0$. Denote the covering space by Z and the non-trivial automorphism of the covering by τ. Since the complex conjugation involution $\operatorname{conj}:\mathbb{C}A\to\mathbb{C}A$ preserves $\mathbb{C}C_1\cup\mathbb{C}C_2$, it can be lifted to Z. One can construct a lifting starting at $x_1\in\mathbb{R}A\setminus(\mathbb{C}C_1\cup\mathbb{C}C_2)$ taking a point $z_1\in Z$ over x_1 and assuming that z_1 is a fixed point for the lifting. After that the lifting is constructed by continuity in a unique way. There are two liftings $c_+,c_-:Y\to Y$ obtained from each other by composition with τ. Assume that $c_+(z_1)=z_1$. Since c_+ is a lifting of conj, the fixed point set of c_+ is projected into the fixed point set $\mathbb{R}A$ of conj. To determine, if a point $x_2 \in \mathbb{R}A \smallsetminus (\mathbb{R}C_1 \cup \mathbb{R}C_2)$ belongs to the image, one has to calculate action of c_+ in the preimage of x_2. Take a point z_2 over x_2. Connect z_1 with z_2 by a path $\tilde{w}$ in the complement of the preimage of branch locus. This path covers a path w which connects x_1 with x_2. Then $c_+(z_2)$ is the end point of the path $c_+ \circ \tilde{w}$. Therefore z_2 is a fixed point of c_+ iff paths $\tilde{w}$ and $c_+ \circ \tilde{w}$ constitute a closed loop. This is equivalent to the condition that the homotopy class of loop $w \operatorname{conj} \circ w$ belongs to the group of the covering. This group consists of homotopy classes of loops in $\mathbb{C}A \smallsetminus (\mathbb{C}C_1 \cup \mathbb{C}C_2)$ unlinked (modulo 2) with $(\mathbb{C}C_1 \cup \mathbb{C}C_2)$. Note that for appropriate choice of s above loops $w \operatorname{conj} \circ w$ and $su_2(\operatorname{conj} \circ s)^{-1}u_1^{-1}$ are homotopic. □ □

4.6 Conversion complex orientations modulo curve into true orientations

Results of the preceding section allow to improve the constructions of Section 4.2 and 4.4. While those constructions give semi-orientation of $\mathbb{R}A \smallsetminus \mathbb{R}C$ which can not be extended over $\mathbb{R}C$, in this section we get a semi-orientation of a two-fold covering space of $\mathbb{R}A$. The role of C will be reduced: the result depends only on the homology class realized by $\mathbb{R}C$.

First, let me remind some classic purely topological constructions. With each codimension 1 closed submanifold Y of a manifold X it is associated a double covering of X. This covering can be constructed in the following way. One cuts X along Y, takes two copies of the result, and glue them to each other identifying a side of the cut of a copy with the opposite side of the cut in the other copy. I will denote the result by $D_Y X$. There is an obvious projection of $D_Y X$ onto X, so $D_Y X$ is a two-fold covering space of X. Various versions of this construction is used extensively in elementary expositions on Riemann surfaces.

The result $D_Y X$ of the construction above depends only on the $\mathbb{Z}_2$-homology class realized by Y in X. In fact, if Y' is another submanifold presenting the same $\mathbb{Z}_2$-homology class as Y, then Y and Y' bound together a domain $H \subset X$, and one can construct a homeomorphism $D_Y X \to D_{Y'} X$ which identifies the copies of H in the copies of $X \smallsetminus Y$ with the copies of H in the copies of $X \smallsetminus Y'$ with the same numbers, and copies of the complementary domain $X \smallsetminus \operatorname{Cl} H$ in the copies of $X \smallsetminus Y$ with the corresponding domains in the copies of $X \smallsetminus Y'$ with distinct numbers. The resulting homeomorphism depends on the choice of H. In the case of connected X it can be chosen in two ways and the corresponding homeomorphisms differs by the non-trivial automorphism of the covering. In the case of disconnected X a choice should be done at each component of X.

The construction will be applied below to the following situation. Let X be the set $\mathbb{R}A$ of real points of a non-singular real algebraic curve C on A. If $H_1(\mathbb{C}A;\mathbb{Z}_2)=0$ then the two-fold covering $D_{\mathbb{R}C}\mathbb{R}A\to\mathbb{R}A$ depends only on homology class $[\mathbb{C}C]\in H_2(\mathbb{C}A;\mathbb{Z}_2)$ realized by $\mathbb{C}C$ in $\mathbb{C}A$. In fact, for any curves C_1, C_2 with $\mathbb{C}C_1$, $\mathbb{C}C_2$ realizing the same element of $H_2(\mathbb{C}A;\mathbb{Z}_2)$ there exists a distinguished pair of homeomorphisms $D_{\mathbb{R}C_1}\mathbb{R}A\to D_{\mathbb{R}C_2}\mathbb{R}A$ which differ from each other by the automorphism of the covering acting non-trivially in each fiber. The homeomorphisms of this pair are related with two domains $H\subset\mathbb{R}A$ which are the images of sets of real points of the two-fold coverings of $\mathbb{C}A$ branched over $\mathbb{R}A_1\cup\mathbb{R}A_2$. Cf. Section 4.5. Since $H_1(\mathbb{C}A;\mathbb{Z}_2)=0$, there is only one two-fold covering of $\mathbb{C}A$ branched over $\mathbb{R}A_1\cup\mathbb{R}A_2$ and the complex conjugation involution of $\mathbb{C}A$ can be lifted in two ways. Each of these two liftings defines its own H.

Returning to the abstract topological situation above, assume that there is an orientation of $X\smallsetminus Y$, which can not be extended across any component of Y. (By the way, it means that Y realizes the $\mathbb{Z}_2$-homology class Poincaré dual to the first Stiefel-Whitney class $w_1(X)$.) Then in the construction above we get in a natural way a semi-orientation of D_YX. To construct it, one should take the orientation of one of the copies of the result of cutting induced by the given orientation of $X\smallsetminus Y$ and take the opposite orientation of the other copy. Together they induce an orientation of D_YX. It is defined up to sign, since it depends of the choice of the first copy. The automorphism of the covering $D_YX\to X$ non-trivial over each point of X reverses these orientations (i. e. sends them to each other). This is also a classical construction known as the construction of the orientation covering for X. The resulting semi-orientation of D_YX is not changed, if one reverses the original orientation of $X\smallsetminus Y$, therefore it depends only on the semi-orientation of $X\smallsetminus Y$.

Let Y' be another submanifold presenting the same $\mathbb{Z}_2$-homology class as Y and H be a domain of X bounded by $Y\cup Y'$. Assume that $X\smallsetminus Y'$ is oriented in such a way that this orientation on H coincide with the orientation above of $X\smallsetminus Y$ and on $X\smallsetminus \operatorname{Cl}H$ is opposite to it. Then the homeomorphisms $D_{Y'}X\to D_YX$ defined by H preserve semi-orientations defined by those orientations of $X\smallsetminus Y'$ and $X\smallsetminus Y$.

Consider now the situation of Section 4.2 and 4.4. Let A be a non-singular real algebraic surface and C be a non-singular real algebraic curve on A. Let $H_1(\mathbb{C}A;\mathbb{Z}_2) = 0$ and $[\mathbb{C}C] + [\mathbb{R}A] = 0 \in H_2(\mathbb{C}A;\mathbb{Z}_2)$. Then the constructions of 4.2 and 4.4 give rise to a complex semi-orientation on $\mathbb{R}A \setminus \mathbb{R}C$ which is not extendible across $\mathbb{R}C$. The construction above associates with it a semi-orientation of $D_{\mathbb{R}C}\mathbb{R}A$. If C' is another non-singular real algebraic curve on A with $\mathbb{C}C'$ realizing the same $\mathbb{Z}_2$-homology class as $\mathbb{C}C$, then the homeomorphisms $D_{\mathbb{R}C}\mathbb{R}A \to D_{\mathbb{R}C'}\mathbb{R}A$ constructed as above preserve the semi-orientation.

In some cases the covering space $D_{\mathbb{R}C}\mathbb{R}A$ is the set of real points of an appropriate algebraic surface. In particular, it happens in the case of projective real algebraic surface of type I_{rel}. If A is such a surface and C is any curve on A which is a transversal intersection of A with a real algebraic hypersurface of odd degree, then $D_{\mathbb{R}C}\mathbb{R}A$ is naturally embedded into the two-fold covering of the ambient projective space $\mathbb{R}P^N$. This covering space is sphere S^N. The projection is a regular map and therefore $D_{\mathbb{R}C}\mathbb{R}A$ is identified with the real part of algebraic variety.

Thus for a projective real algebraic surface of type I_{rel} the complex semi-orientation modulo hyperplane section corresponds to some semi-orientation of the real part of another real algebraic surface: its preimage under covering $S^N \to \mathbb{R}P^N$.

5 Conclusion. Survey of some subsequent results

In this section I mention shortly various further developments. I plan to give a detailed presentation of them elsewhere.

5.1 Complex orientations of high-dimensional varieties

Definition for complex orientations of a surface bounding in complexification which is given above in Section 4.4 provides not only opportunity to understand behavior of complex orientations modulo changing curve, presented in Section 4.5. It suggests a way for generalizing of the notion of complex orientations in two directions: for high-dimensional varieties and for high-dimensional analogs of orientation.

To begin with, consider lower-dimensional case: reformulate the definition of complex orientations of curves in spirit of Section 4.4. Let A be a non-singular curve of type I. Given points x_1, $x_2 \in \mathbb{R}A$, we have to give a criteria for local orientations of $\mathbb{R}A$ at x_1 and x_2, if these local orientations agree with the same complex orientation of A. Denote by y_i a point of the boundary of a tubular neighborhood of $\mathbb{R}A$ in $\mathbb{C}A$ obtained from x_i by a shift in direction of the normal vector which is $\sqrt{-1}$ times a tangent vector directed according to the local orientation of $\mathbb{R}A$. The pair y_1, y_2 is a 0-cycle $\mathbb{Z}_2$-homologous to zero. If its linking number with $\mathbb{R}A$ in $\mathbb{C}A$ is zero $\pmod 2$, then the local orientations agree with the same complex orientation of A, otherwise the local orientations do not agree with any complex orientation of A.

In high-dimensional case, one may do a similar process. Let A be a non-singular n-dimensional real algebraic variety bounding in complexification. Take points x_1, $x_2 \in \mathbb{R}A$ equipped with local orientations of $\mathbb{R}A$. Assume that these local orientations are defined by bases $e^1_1, \dots, e^1_n$ and $e^2_1, \dots, e^2_n$ of tangent spaces of $\mathbb{R}A$ at x_1, x_2. Fix a tubular neighborhood of $\mathbb{R}A$ in $\mathbb{C}A$ and denote the fiber of it over x_i by D_i and the boundary of D_i by S_i.

Let y_i be the point of S_i obtained by shift of x_i in direction of $\sqrt{-1}e^i_1$. Choose a path s connecting y_1 with y_2 in $\mathbb{C}A \smallsetminus \mathbb{R}A$. It is possible provided $\dim A > 1$. In sphere S_i connect antipodal points y_i and $\operatorname{conj}(y_i)$ by the meridian passing through the point obtained from x_i by a shift in direction of $\sqrt{-1}e^i_2$. Those meridians together with s and $\operatorname{conj}\circ s$ make a 1-cycle c^1. If $n = 2$, we have to consider the linking number of this cycle with $\mathbb{R}A$ in $\mathbb{C}A$, as in Section 4.5. Otherwise this cycle is $\mathbb{Z}_2$-homologous to zero, provided $H_1(\mathbb{C}A; \mathbb{Z}_2) = 0$. Moreover, if $\pi_1(\mathbb{C}A) = 0$, it bounds a disk in $\mathbb{C}A \smallsetminus \mathbb{R}A$. Take this disk, or a chain s^1 in $\mathbb{C}A \smallsetminus \mathbb{R}A$ with $\partial s^2 = c^1$. Consider $s^2 + \operatorname{conj}(s^2)$. This is a 2-chain whose boundary consists of great circles of S_1 and S_2. Fill this boundary with geodesic disks in S_i passing through points obtained from x_i by shifts in directions of $\sqrt{-1}e^i_3$. Denote the resulting 2-cycle by c^2. In the case $n = 3$ we have to consider its linking number with $\mathbb{R}A$. If this linking number is zero, then the local orientations are announced to agree with the same complex orientation, otherwise the local orientations do not agree with any complex orientation. If $n > 3$, then the process should be continued.

I do not mean to discuss here conditions under which this construction gives a well-defined complex orientation. This question is far from being trivial. It admits, however, a simple solution in the case of affine varieties. Details will be given elsewhere.

5.2 *Spin*-structure of a real algebraic surface bounding in complexification

Remind that a *Spin*-structure of a manifold is a reduction of the structure group of its tangent bundle to group *Spin*. It can be described in more homological terms as follows: *Spin*-structure is equivalent to a pair consisting of an orientation and a $\mathbb{Z}_2$-valued functional defined on the set of framed loops. By a framed loop in n-dimensional manifold X I mean here a map $l : S^1 \to X$ equipped with a continuous field of $(n-1)$-frames assigning to each $t \in S^1$ a sequence of $n-1$ linear independent tangent vector of X at point $l(t)$. The functional should satisfy two conditions: first, it should take equal values on homological framed loops; second, it takes non-trivial value on any framed loop with constant map $l : S^1 \to X$ and framing defining non-contractible loop in the space of all $(n-1)$-frames of $T_{l(S^1)}X$. In the case of two dimensional X the framings are vector fields.

Let A be a non-singular real algebraic surface bounding in complexification. The following construction gives a *Spin*-structure. For the orientation of $\mathbb{R}A$ we take one of the complex orientations defined in Section 3.2 above. Now let $l : S^1 \to \mathbb{R}A$ be a loop equipped with a vector field. Shift l from $\mathbb{R}A$ to $\mathbb{C}A \smallsetminus \mathbb{R}A$ along this vector field multiplied by $\sqrt{-1}$ and take linking number of the resulting loop with $\mathbb{R}A$. It is easy to check that this construction gives a functional on the set of framed loops satisfying the conditions above.

This construction admits generalizations to high-dimensional case similar to constructions mentioned in Section 5.1.

Let me remind that in the 2-dimensional case *Spin*-structures admit also a description in terms of Z_2-valued quadratic forms on one-dimensional homology. Given a closed orientable surface F, by a $\mathbb{Z}_2$-valued quadratic form on $H_1(F; \mathbb{Z}_2)$ one means a mapping $q : H_1(F; \mathbb{Z}_2) \to \mathbb{Z}_2$ such that $q(x+y) = q(x) + q(y) + x \circ y$ for any $x, y \in H_(F; \mathbb{Z}_2)$, where $x \circ y$ denotes the intersection number of classes x and y. There is a one to one correspondence between $\mathbb{Z}_2$-valued quadratic forms on $H_1(F; \mathbb{Z}_2)$ and *Spin*-structures on F with a fixed orientation of F, see e.g. [5]. The quadratic form corresponding to a *Spin*-structure assings to the homology class realized by a collection of disjoint simple closed loops the number of those loops (modulo 2) plus the sum of values of the *Spin*-structure on the loops equipped with a vector field consisting of non-zero vectors tangent to the loops.

Therefore, for any non-singular real algebraic surface A which bounds in complexification, there is a natural quadratic form $q : H_1(\mathbb{R}A; \mathbb{Z}_2) \to \mathbb{Z}_2$

which assigns to the class represented by a collection of disjoint simple closed loops $l_1, \ldots, l_k$ the number k of those loops (modulo 2) plus the sum of the linking numbers of $\mathbb{R}A$ and the loops obtained from l_i by a small shift in the direction of a vector field obtained from a field of vectors tangent to l_i by mutplication by $\sqrt{-1}$.

Note, that the construction does not involve conj. Therefore it makes sense in situations when conj does not exist. For example, in the situation of a Lagrangian surface homological modulo 2 to zero in a symplectic 4-manifold. The Spin-structure has been used in study of Lagrangian tori in $\mathbb{C}^2$, initiated by L. Polterovich.

5.3 Pin^--structure of a real algebraic surface of type I_{rel}

The set of real points of a real algebraic surface of type I_{rel} may be non-orientable. The obvious modification of the construction of Section 5.2 gives a *Spin*-structure on the complement of any hyperplane section of real algebraic surface of type I_{rel}. But this can be essentially improved.

For non-orientable surfaces, there is an analog of *Spin*-structure which is called Pin^--structure. It can be presented as $\mathbb{Z}_4$-valued quadratic form on one-dimensional $\mathbb{Z}_2$-homology of the surface. Given a closed surface F, by a $\mathbb{Z}_4$-valued quadratic form on $H_1(F;\mathbb{Z}_2)$ one means a mapping $q : H_1(F;\mathbb{Z}_2) \to \mathbb{Z}_4$ such that $q(x+y) = q(x) + q(y) + 2(x \circ y)$ for any $x, y \in H_1(F;\mathbb{Z}_2)$ where $x \circ y$ denotes an intersection number (taking value in $\mathbb{Z}_2$ as above) and 2 denotes the standard inclusion $\mathbb{Z}_2 \to \mathbb{Z}_4$.

To define such a form for the set of real points of a non-singular real algebraic surface A of type I_{rel}, consider a collection of disjoint embedded loops $l_1, \ldots, l_k$ which presents the homology class, for which we want to define the value of our quadratic form. Consider a hyperplane section C with $\mathbb{R}C$ transversal to the loops. For each of l_i take a non-zero tangent vector field, multiply it by $\sqrt{-1}$, shift l_i along the result and denote the linking number of $\mathbb{R}A$ with the loop obtained by λ_i. Note that $\lambda_i \in \mathbb{Z}_2$ and $2\lambda_i \in \mathbb{Z}_4$.

The value of the quadratic form on the class is equal to $2\sum_{i=1}^k \lambda_i$ plus $2k \mod 4$ plus the number of intersection points of $\mathbb{R}C$ with $\bigcup_{i=1}^k l_i$ reduced modulo 4.

One may check that this rule gives a well defined result and that it is a

$\mathbb{Z}_4$-valued quadratic form.

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