MUTUAL POSITION OF HYPERSURFACES IN PROJECTIVE SPACE

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Abstract. In this paper elementary characteristics for mutual positions of several disjoint closed smooth hypersurfaces in a projective space are studied. In terms of these characteristics, new restrictions on topology of real algebraic hypersurfaces of a given degree are formulated.

There is a small simple fragment, which appears repeatedly in papers on the topology of real plane projective algebraic curves. This is a purely topological description of possible mutual positions of several disjoint circles in the projective plane. Since one of the main problems on the topology of real plane projective algebraic curves is what topological pictures are realized by curves of a given degree, it is necessary to fix first terms in which these pictures can be discussed.

The goal of this paper is to generalize this to the case of hypersurfaces of a projective space.

In the case of plane projective curves the whole topology can be described in homological terms (although they are not called in this way, being simpler than almost everything related to homology): a circle can be positioned in the projective plane one- or two-sidedly and a two-sided circle can encircle another one. In higher dimensions there is topology which cannot be expressed in homology. This happens even in the next dimension: handles of surfaces in $\mathbb{R}P^3$ can be knotted. However it makes sense to look first at the homological part of the story, since it catches the simplest and roughest phenomena. By the way, in the knot theory, which is considered a model for studying differences between embeddings, the homology part appears only after auxiliary geometric construction. Probably, this is why it is usually underestimated. However even a lean homology of the projective space provides additional ways of linking and knotting surfaces in the projective space, and makes the story more complicated than a similar story for surfaces in Euclidean space.

A connected hypersurface supports a part of homology of the projective space. This part is described by a single number called the rank of the hypersurface. There are restrictions on mutual position of disjoint hypersurfaces and lines meeting them formulated in terms of the ranks. Connected components of the complement of several disjoint hypersurfaces give rise to the graph of adjacency, which is a tree,

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and contains subtrees defined in a natural way. This approach to the topological situation pays off: it provides terms for formulations of new restrictions on the topology of real algebraic hypersurfaces of a given degree in a projective space.

I begin with facts, which may be considered well-known. These facts are discussed with details because they are related to specific features of the projective space. It occupies the first two sections. In Section 3 various graphs describing the mutual position of connected components of a hypersurface in the projective space are introduced. Section 4 contains applications to topology of real algebraic hypersurfaces. These applications are simple corollaries of the Bézout theorem, but their formulations become possible due to the definitions of Section 3. In Section 5 the upper bound of the number of noncontractible components of a real algebraic surface of a given degree in $\mathbb{R}P^3$ are discussed.

1. Connected Hypersurfaces

1.1. Two-Sided and One-Sided Hypersurfaces. For brevity, we shall refer to smooth closed two-dimensional submanifolds of the real projective space $\mathbb{R}P^3$ as *surfaces* when there is no danger of confusion. We shall consider also an immediate generalization: smooth closed $(n-1)$-dimensional submanifolds of $\mathbb{R}P^n$. They will be referred to as *hypersurfaces* of $\mathbb{R}P^n$.

Since the homology group $H_{n-1}(\mathbb{R}P^n; \mathbb{Z}_2)$ is $\mathbb{Z}_2$, a connected hypersurface can be situated in $\mathbb{R}P^n$ in two ways: either zero-homologous, or realizing the nontrivial homology class.

In the first case, the hypersurface divides the projective space into two connected domains, being the boundary for both of them. Hence, the hypersurface divides its tubular neighborhood, i.e. is *two-sided.*

In the second case, the complement of the hypersurface in the projective space is connected. (Indeed, if it were not connected, the hypersurface would bound and thereby realize the zero homology class.)

Moreover, a non-zero-homologous hypersurface is *one-sided,* i.e., does not separate even its tubular neighborhood.

This can be shown in many ways. For example, if the hypersurface were two-sided and its complement were connected, there would exist a nontrivial infinite cyclic covering of $\mathbb{R}P^n$, which would contradict the fact that $\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$. The infinite cyclic covering could be constructed by gluing an infinite sequence of copies of $\mathbb{R}P^n$ cut along the surface: each copy has to be glued along one of the sides of the cut to the other side of the cut in the next copy.

1.2. Orientability of Hypersurface. *A connected two-sided hypersurface in* $\mathbb{R}P^{2n-1}$ *is orientable,* since it bounds a part of the ambient space, which is orientable.

Therefore a connected two-sided surface in $\mathbb{R}P^3$ is homeomorphic to sphere or sphere with handles. There is no restriction on the number of handles: one can take an embedded sphere bounding a small ball, and adjoin to it any number of handles.

A one-sided even-dimensional hypersurface is nonorientable. Indeed, its normal bundle is nonorientable, while the restriction of the tangent bundle of $\mathbb{R}P^n$ to the hypersurface is orientable (since an odd-dimensional projective space is). The restriction of the tangent bundle of $\mathbb{R}P^n$ to the hypersurface is the Whitney sum of the normal and tangent bundles of the hypersurface. Therefore it cannot happen that only one of these three bundles is not orientable.

A one-sided odd-dimensional hypersurface is orientable. Indeed, it realizes the nontrivial homology class, which is dual to the first Stiefel-Whitney class (i.e., there is an orientation of the complement which jumps as one passes through the hypersurface), and any hypersurfaces realizing the homology class dual to the first Stiefel-Whitney class is orientable (the orientation of the complement induces from the both sides of the hypersurface the same orientation).

A two-sided odd-dimensional hypersurface is nonorientable, unless it contains no loop noncontractible in the ambient space. Indeed, if such a loop exists, it is disorienting in the ambient space, and the restriction of the normal bundle to it is trivial, for the hypersurface is two-sided. Therefore, the loop is disorienting for the hypersurface. On any loop contractible in the ambient space all the three bundles are orientable.

1.3. Topological Type of One-Sided Surface. Contrary to the case of two-sided surfaces, in the case of one-sided surfaces there is an additional restriction on their topological types, which is, in fact, a counter-part for orientability of two-sided surfaces.

1.3.A. *The Euler characteristic of a connected one-sided surface in $\mathbb{R}P^3$ is odd.*

In particular, it is impossible to embed a Klein bottle to $\mathbb{R}P^3$. (The Euler characteristic of a connected two-side surfaced in $\mathbb{R}P^3$ is even. However, this is not an additional restriction on its topology, but follows from orientability: the Euler characteristic of any closed oriented surface is even.) By topological classification of closed surfaces, a nonorientable connected surface with odd Euler characteristic is homeomorphic to the projective plane or to the projective plane with handles. Any surface of this sort can be embedded into $\mathbb{R}P^3$: for the projective plane $\mathbb{R}P^3$ is the native ambient space, and one can adjoin to it in $\mathbb{R}P^3$ any number of handles.

Proof of 1.3.A. Let S be a connected one-sided surface in $\mathbb{R}P^3$. By a small shift, it can be made transversal to the projective plane $\mathbb{R}P^2$ standardly embedded into $\mathbb{R}P^3$. Since both surfaces are one-sided, they realize the same homology class in $\mathbb{R}P^3$. Therefore their union bounds in $\mathbb{R}P^3$: one can color the complement $\mathbb{R}P^3 \smallsetminus (S \cup \mathbb{R}P^2)$ into two colors in such a way that the components adjacent from the different sides to the same (two-dimensional) piece of $S \cup \mathbb{R}P^2$ would be of different colors. It is a kind of checkerboard coloring.

Consider the disjoint sum Q of the closures of those components of $\mathbb{R}P^3 \smallsetminus (S \cup \mathbb{R}P^2)$ which are colored with the same color. It is a compact 3-manifold. It is oriented since each of the components inherits orientation from $\mathbb{R}P^3$. The boundary of this 3-manifold is composed of pieces of S and $\mathbb{R}P^2$. It can be thought of as the result of cutting both surfaces along their intersection curve and regluing. The intersection curve is replaced by its two copies, while the rest of S and $\mathbb{R}P^2$ does not change. Since the intersection curve consists of circles, its Euler characteristic is zero. Therefore $\chi(\partial Q) = \chi(S) + \chi(\mathbb{R}P^2) = \chi(S) + 1$. On the other hand, $\chi(\partial Q)$ is even since ∂Q is a closed oriented surface (∂Q inherits an orientation from Q). Thus $\chi(S)$ is odd. $\square$

This proof works in a more general setup giving rise to the following theorem.

1.3.B. *The Euler characteristic of a connected one-sided hypersurface in* $\mathbb{R}P^{2n+1}$ *is odd.* $\square$

1.4. Contractibility. A one-sided connected surface in $\mathbb{R}P^3$ contains a loop which is not contractible in $\mathbb{R}P^3$. Such a loop can be detected in the following way: Consider the intersection of the surface with any one-sided transversal surface (e. g., $\mathbb{R}P^2$ or a surface obtained from the original one by a small shift). The homology class of the intersection curve is the self-intersection of the nonzero element of $H_2(\mathbb{R}P^3\,;\mathbb{Z}_2)$. Since the self-intersection is the nonzero element of $H_1(\mathbb{R}P^3\,;\mathbb{Z}_2)$, the intersection curve contains a component noncontractible in $\mathbb{R}P^3$.

A two-sided connected surface in $\mathbb{R}P^3$ can contain no loops noncontractible in $\mathbb{R}P^3$ (this happens, for instance, if the surface lies in an affine part of $\mathbb{R}P^3$). Of course, if a surface contains a loop noncontractible in $\mathbb{R}P^3$, the surface is not contractible in $\mathbb{R}P^3$ itself. Moreover, then it meets any one-sided surface, since the noncontractible loop realizes the nonzero element of $H_1(\mathbb{R}P^3\,;\mathbb{Z}_2)$ and this element has nonzero intersection number with the homology class realized by a one-sided surface.

If any loop on a connected surface S embedded in $\mathbb{R}P^3$ is contractible in $\mathbb{R}P^3$ (which means that the embedding homomorphism $\pi_1(S) \to \pi_1(\mathbb{R}P^3)$ is trivial), then there is no obstruction to contract the embedding, i.e., to construct a homotopy between the embedding $S \to \mathbb{R}P^3$ and a constant map. One can take a cell decomposition of S, contract the 1-skeleton (extending the homotopy to the whole S), and then contract the map of the 2-cell, which is possible, since $\pi_2(\mathbb{R}P^3) = 0$. A surface of this sort is called *contractible (in* $\mathbb{R}P^3$*)*.

1.4.A *Remark.* It may happen, however, that there is no isotopy relating the embedding of a contractible surface to a map to an affine part of $\mathbb{R}P^3$. The simplest example of a contractible torus which cannot be moved by an isotopy to an affine part of $\mathbb{R}P^3$ is shown in Figure 1. See Drobotukhina's paper [3], where the corresponding problem for knots in $\mathbb{R}P^3$ is discussed.

1.5. Rank of Hypersurface. The division of two-sided surfaces into contractible and noncontractible ones can be thought of as a refinement of the division of all the surfaces into one- and two-sided. Indeed, one-sided surfaces can be characterized as those for which the inclusion induces nontrivial homomorphisms in homology of $\mathbb{Z}_2$ coefficients in dimensions ≤ 2; two-sided noncontractible surfaces are those for which the inclusion homomorphism in dimension 2 is trivial, while in dimensions ≤ 1 it is not; for two-sided contractible surfaces only the inclusion homomorphism in dimension 0 is not trivial.

Define the *rank* of a hypersurface to be the maximal integer r such that the homomorphism induced in r-dimensional homology (with $\mathbb{Z}_2$ coefficients) by the inclusion of the hypersurface into the projective space is not trivial. Notice that

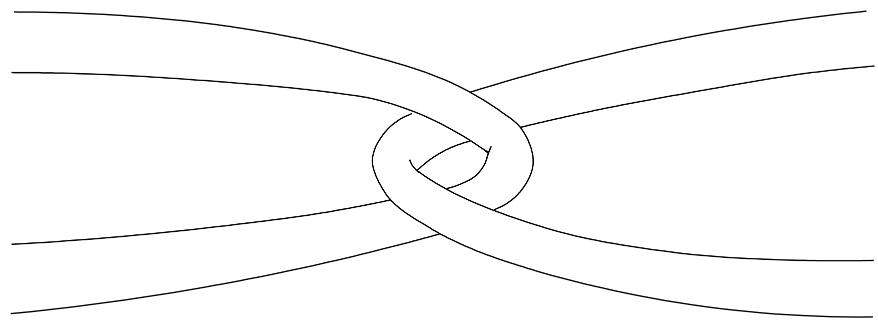

FIGURE 1. An affine part of a torus embedded into the three-dimensional projective space in such a way that the embedding is homotopic to a constant map, but not isotopic to an embedding with image fitting in an affine part of the space.

a hypersurface is one-sided iff its rank is equal to its dimension, and a two-sided surface is not contractible iff its rank is 1.

A hypersurface S of rank r in $\mathbb{R}P^n$ intersects any projective subspace of dimension $\geq n-r$. This follows from the structure of the intersection ring of the projective space. Indeed, the nontrivial class of dimension r has nontrivial intersection with all the nontrivial classes of dimension $\geq n-r$ in $H_*(\mathbb{R}P^n)$. This argument proves more, namely: *the intersection of a hypersurface of rank r in $\mathbb{R}P^n$ with a transversal projective subspace L of dimension $k \geq n-r$ is a hypersurface of L with rank $r-n+k$.*

The rank is involved in many restrictions on topology of real algebraic hypersurfaces of a given degree. See Kharlamov [11], Nikulin [14]

1.6. Complement. As it was stated above, the complement $\mathbb{R}P^n \smallsetminus S$ of a connected hypersurface S two-sidedly embedded in $\mathbb{R}P^n$ consists of two connected components. If S has rank r $(< n-1)$, then the inclusion homomorphisms for both components to the ambient space are not trivial up to dimension r. Indeed, a cycle on S non-zero-homologous in the ambient space can be pushed to each of the components, since S is two-sided.

In the higher dimensions only one component contains a cycle non-homologous to zero in the ambient space. This follows from exactness of the Mayer-Vietoris sequence. Indeed, if both components had non-zero-homologous cycles, these cycles would not be homologous in the union: this would happen only if the intersection had a nontrivial homology class mapped by the inclusion homomorphisms to the classes in the components. But in each dimension the homology group of the projective space contains only one nontrivial element.

All the higher-dimensional homology classes are located in the same component, since all of them are realized by the intersections of projective subspaces with the cycle realizing the $(n-1)$-dimensional class and located in one of the components.

The component of the complement that supports the nontrivial high-dimensional homology classes of the projective space is said to be *exterior* and the other one *interior.* This distinction does not exist for two-sided hypersurfaces of the maximal rank (=$n-1$). In this case the components may be positioned in the same way.

The simplest example of this situation is provided by a one-sheeted hyperboloid in $\mathbb{R}P^3$. It is homeomorphic to torus and its complement consists of two solid tori. So, this is a Heegaard decomposition of $\mathbb{R}P^3$. There exists an isotopy of $\mathbb{R}P^3$ made of projective transformation exchanging the components. (One-sheeted hyperboloid can be moved in $\mathbb{R}P^3$ to a position, in which it intersects the affine space in a hyperbolic paraboloid defined by equation $xy = z$. The hyperbolic paraboloid is invariant under rotation by π around the axes OX (i.e., transformation $(x, y, z) \mapsto (x, -y, -z)$), which exchanges the components of the complement.)

A connected surface decomposing $\mathbb{R}P^3$ into two handlebodies is called a *Heegaard surface.* Heegaard surfaces are the most unknotted surfaces among two-sided noncontractible connected surfaces. They may be thought of as unknotted noncontractible surfaces.

A contractible connected surface S in $\mathbb{R}P^3$ is said to be *unknotted,* if it is contained in a ball B embedded into $\mathbb{R}P^3$ and divides this ball into a ball with handles (which is the interior of S) and a ball with handles with an open ball deleted. Any two unknotted contractible surfaces of the same genus are ambiently isotopic in $\mathbb{R}P^3$. Indeed, first the balls containing them can be identified by an ambient isotopy (see, e. g., Hirsch [8], Section 8.3), then it follows from uniqueness of Heegaard decomposition of sphere that there is an orientation preserving homeomorphism of the ball mapping one of the surfaces to the other. Any orientation preserving homeomorphism of a 3-ball is isotopic to the identity.

2. Linking Numbers

2.1. Linking Numbers of Cycles. Recall the definitions and basic facts related to linking numbers. *Linking number* is a classical term, which has numerous meanings, even in the context of this paper. All of them are based on the following geometric construction.

Let a and b be disjoint cycles in $\mathbb{R}P^n$ with integer coefficients of dimensions p and q , respectively, with $p + q = n - 1$. Take a rational chain c transversal to a and having boundary $\partial c = b$. The intersection number of a and c is the linking number $\operatorname{lk}(a, b)$ of a and b. It does not depend on the choice of c. Of course, if b is homologous to zero as a cycle with integer coefficients, then c can be taken with integer coefficients and the intersection number is an integer. Otherwise it may be a half-integer. For example, the linking number of two skew lines in $\mathbb{R}P^3$ is $\pm\frac{1}{2}$. In this case for c one can take a (projective) plane containing b with coefficient $\pm\frac{1}{2}$.

This example can be generalized to the case of arbitrary smooth b as follows: choose a point p in $\mathbb{R}P^n \setminus b$ and construct a cone C over b with vertex p. This is a cone in the *projective* sense, i.e., the union of projective lines passing through p and points of b. As in the case of line, it is adjacent to b from two sides. For creating c, it must be equipped with the appropriate orientation and multiplicity $\frac{1}{2}$.

2.2. Linking Number via Counting Lines. This c gives rise to the following rule for calculating $\mathrm{lk}(a, b)$ for disjoint smooth closed cycles a, b, cf. [3]. Fix a point $p \in \mathbb{R}P^n \smallsetminus (a \cup b)$. By a choice of p one can meet the following conditions:

- the number of lines passing through p and intersecting a and b is finite,
- each of these lines meets each of a and b in a single point,
- each of these lines is not contained in the tangent planes to a and b at the intersection of the line with a and b.

Let l be one of the lines connecting a and b and passing through p, and let a_0, b_0 be its intersection points with l. Choose one of the two segments of l with end points a_0, b_0 and denote it by s. Let $V = \{v_1, \dots, v_p\}$ and $W = \{w_1, \dots, w_q\}$ be the bases of tangent spaces of a and b at a_0 and b_0, respectively, which are positive with respect to the orientations of a and b. Let L be a vector tangent to l at a_0 and directed inside s. Let $W' = \{w'_1, \dots, w'_q\}$ be a sequence of vectors at a_0 such that w'_i is tangent to the plane containing l and w_i and directed to the same side of s as w_i in an affine part of the plane containing s and w_i. See Figure 2. The vectors of V, L, W' comprise a basis of the tangent space $T_{a_0}\mathbb{R}P^n$. The value taken by the orientation of $\mathbb{R}P^n$ on this frame is associated with l. One can easily check that:

The sum of the values associated with all the lines passing through p and meeting a, b is $2\,\mathrm{lk}(a, b)$.

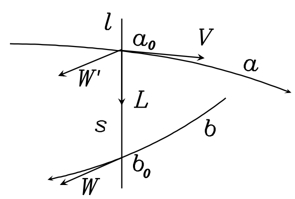

FIGURE 2. Construction of the frame V, L, W'.

2.3. Linking Numbers in Homology of $\mathbb{R}P^n$. The linking number defined above does depend on the cycles, for various disjoint representatives of their homology classes the linking number varies. However, it does not change modulo 1 and defines a bilinear form $H_p(\mathbb{R}P^n) \times H_q(\mathbb{R}P^n) \to \mathbb{Q}/\mathbb{Z}$. The value of this form on classes $\alpha \in H_p(\mathbb{R}P^n)$ and $\beta \in H_q(\mathbb{R}P^n)$ is called the *linking number* of α, β and denoted by $\mathrm{lk}(\alpha, \beta)$. Of course, the linking number of $0 \in H_p(\mathbb{R}P^n)$ with any class is 0, while the linking number of the only non-trivial classes is $\frac{1}{2}$.

2.4. Linking Pairing Between Subsets. One can catch more homological sense of the geometric construction above by consideration disjoint sets $A, B \subset \mathbb{R}P^n$ containing cycles a, b respectively. The linking number of the cycles a, b (in $\mathbb{R}P^n$) depends only on the classes $\alpha \in H_p(A)$, $\beta \in H_q(B)$. It is called the *linking number* of α and β and denoted by $\mathrm{lk}(\alpha, \beta)$. The corresponding map

$$\mathrm{lk} : H_p(A) \times H_q(B) \to \mathbb{Z}[1/2]$$

is bilinear and called *linking pairing.*

It can be defined in more modern terms of the intersection number pairing

$$H_p(\mathbb{R}P^n \smallsetminus B) \times H_{q+1}(\mathbb{R}P^n, B; \mathbb{Q}) \to \mathbb{Q}$$

of the Alexander-Pontryagin duality as the intersection number of the image of α under the inclusion homomorphism $H_p(A) \to H_p(\mathbb{R}P^n \smallsetminus B)$ with the rational homology class $\gamma \in H_{q+1}(\mathbb{R}P^n, B; \mathbb{Q})$ whose image under the differential

$$H_{q+1}(\mathbb{R}P^n, B; \mathbb{Q}) \to H_q(B; \mathbb{Q})$$

of the rational homology sequence of pair $(\mathbb{R}P^n, B)$ is equal to the image of β under the coefficient homomorphism $H_q(B) \to H_q(B; \mathbb{Q})$.

2.4.A. *Let A, B be disjoint subsets of $\mathbb{R}P^3$. If the linking pairing* $\mathrm{lk} : H_p(A) \times H_q(B) \to \mathbb{Z}[\frac{1}{2}]$ *is not trivial, then for any point $P \in \mathbb{R}P^n$ there exists a line passing through P and intersecting A and B.*

Proof. Indeed, for any cycles $a \subset A$, $b \subset B$ with $\mathrm{lk}(a, b) \neq 0$ there exists a line passing through P and meeting a and b. $\square$

2.4.B Corollary. *If A, B are disjoint sets such that the inclusion homomorphisms $H_p(A) \to H_1(\mathbb{R}P^n)$ and $H_q(B) \to H_q(\mathbb{R}P^n)$ are not trivial, then for any point $P \in \mathbb{R}P^n$ there exists a line passing through P and intersecting A and B.*

3. Mutual Position of Components

3.1. In Presence of One-Sided Component. At most one connected component of a (closed) hypersurface in $\mathbb{R}P^n$ may be one-sided. Moreover, if a hypersurface has a one-sided component then all other components are of rank 0. Indeed, one-sided connected hypersurface realizes the nonzero element of $H_{n-1}(\mathbb{R}P^n ; \mathbb{Z}_2)$, which intersects any nonzero homology class of positive dimension in $\mathbb{R}P^n$.

The components of rank 0 are naturally ordered: a component can contain other component in its interior and this gives rise to a partial order in the set of components of rank 0. If the interior of A contains B, then one says that A *envelopes* B.

3.2. Main Components. This is easy to generalize. At most one component of a hypersurface in $\mathbb{R}P^n$ may be of rank $r > n/2$. If such a component exists, it is called the *main component.* The rank of any other component is at most $n - r - 1$. The components different from the main one are ordered as above. (The rank of any one of them is less than $n - 1$, hence its complement consists of interior and exterior components.)

A special situation can appear in the case of a surface in $\mathbb{R}P^3$. If it contains no one-sided component, it may contain several components of rank 1 (i.e., two-sided noncontractible components.) Since each of them divides the projective space into halves, which are homology equivalent, here is no way to say that one of them envelopes another one. All the components of rank 1 are said to be *main.*

3.3. Region Trees. The connected components of a hypersurface in $\mathbb{R}P^n$ divide $\mathbb{R}P^n$ into connected regions. Let us construct a graph of adjacency of these regions: assign a vertex to each of the regions and connect two regions with an edge if the corresponding regions are adjacent to the same two-sided component of the hypersurface. Since the projective space is connected and its fundamental group is finite, the graph is contractible, i.e., it is a tree. It is called the *region tree* of the hypersurface.

For a region we define the *rank* exactly as for a connected hypersurface (cf. 1.5): this is the maximal integer r such that the homomorphism induced in homology with $\mathbb{Z}_2$ coefficients of dimension r by the inclusion of the region to the projective space is not trivial.

The regions of rank $\geq k$ and components of the hypersurface of rank $\geq k$ comprise a subtree of the region tree. This follows from the fact that each $H_k(\mathbb{R}P^n\,;\mathbb{Z}_2)$ is $\mathbb{Z}_2$ via consideration of the Mayer-Vietoris sequence, as above in Section 1.6, when we proved that only one connected component of a connected two-sided hypersurface of rank k can support a nontrivial homology class of $\mathbb{R}P^n$ of dimension $> k$.

This subtree is called the *rank k region tree.* It is obviously isomorphic to the region tree of the hypersurface composed of the components of the original hypersurface having rank $\geq k$.

3.3.A. *Let A and B be regions of ranks k and $n-k-d-1$ respectively for a hypersurface in $\mathbb{R}P^n$. Then for any projective subspace P of dimension d there exists a projective subspace of dimension $d+1$ containing P and intersecting A and B.*

Proof. Let a be a k-dimensional cycle in A and b a $(n-k-d-1)$-dimensional cycle in B realizing nonzero $\mathbb{Z}_2$-homology classes of $\mathbb{R}P^n$. If P intersects a, then take a point in b and consider the cone over P centered at this point. This is obviously a desired space.

If P and a are disjoint, consider the join of a and P (i.e., the union of all the lines joining a and P). This is a cycle of dimension $k+1+d$, which can be obtained by taking $d+1$ times a cone: the first time, the cone over a with a vertex in P, then the cone over this cone with vertex at other point of P, etc. Each time we get a cycle of the next dimension realizing a nonzero homology class. The final one intersects b since the nonzero classes of dimensions $k+1+d$ and $n-k-d-1$ in $\mathbb{R}P^n$ has nonzero intersection number. $\square$

3.3.B Corollary: Apparently Intersecting Regions. *Let A and B be regions of ranks k and $n-k-1$ respectively for a hypersurface in $\mathbb{R}P^n$. Then for any point P there exists a line passing through P and intersecting A and B.*

The direct proof of 3.3.B contained in the proof of 3.3.A resembles the linking number arguments. Indeed, there is a similar way to use more delicate linking numbers. The necessary notions are prepared in the following section. I do not know a similar refinement for 3.3.A.

3.4. Trees of Linked Regions.

3.4.A. *Let p, q, n be integers with $p + q = n - 1$. Consider a hypersurface in $\mathbb{R}P^n$ and the set of regions B which have non-trivial linking pairing $H_p(A) \times H_q(B) \to \mathbb{Z}[\frac{1}{2}]$ with a given region A. In the region tree they comprise together with A the set of vertices of a subtree.*

Proof. Let B be a region such that the linking pairing

$$\mathrm{lk} : H_p(A) \times H_q(B) \to \mathbb{Z}[\tfrac{1}{2}] \tag{1}$$

is not trivial and R be a region which corresponds to a vertex of the region tree positioned between the vertices corresponding to A and B. Since the linking pairing (1) is not trivial, there exists a cycle b in B such that any chain c in $\mathbb{R}P^n$ with $\partial c = b$ has a non-zero intersection number with some cycle in A.

Since R separates A from B in $\mathbb{R}P^n$, $c \cap R$ separates $\partial c = b$ from $c \cap a$. Take a cycle d in $c \cap R$ separating $\partial c = b$ from $c \cap a$. This is a cycle in R non-trivially linked with b. $\square$

The subtree of the region tree described in 3.4.A is called the *tree of regions (p, q)-linked with A.*

3.4.B *Remark.* If $n = 3$ and A is a noncontractible region then the tree of regions $(1, 1)$-linked with A contains all the vertices corresponding to noncontractible regions, and hence the domain tree.

4. Restrictions on Topology of Real Algebraic Hypersurfaces

4.1. Nonsingular Real Algebraic Hypersurfaces. Recall basic definitions related to real algebraic hypersurfaces. By a *real algebraic hypersurface of degree m in the n-dimensional projective space* we shall mean a real homogeneous polynomial of degree m in $n + 1$ variables considered up to a constant factor. A *real point* of a real algebraic hypersurface represented by a polynomial F is a point $(x_0 : x_1 : \cdots : x_n) \in \mathbb{R}P^n$ such that $F(x_0, x_1, \ldots, x_n) = 0$. Similarly one defines complex point $(x_0 : x_1 : \cdots : x_n) \in \mathbb{C}P^n$ of a real algebraic hypersurface. The set of real points of a real algebraic hypersurface A is denoted by $\mathbb{R}A$, the set of complex points is denoted by $\mathbb{C}A$. A point $(x_0 : x_1 : \cdots : x_n)$ of the hypersurface represented by a polynomial F is called a *singular* point of the surface if all the partial derivatives of F vanish at $(x_0, x_1, \ldots, x_n)$. A hypersurface is said to be *nonsingular* if it has no singular points (neither real nor complex). Real algebraic hypersurfaces of degree m comprise a space, which is a real projective space. Nonsingular surfaces make open dense subset of this space, hence they have to be studied first. Here is the coarsest classification topological problem about them.

***4.1.A* Topological Classification Problem.** *Up to homeomorphism, what are the possible sets of real points of a nonsingular real projective algebraic hypersurface of degree m in $\mathbb{R}P^n$?*

A more refined problem:

4.1.B Ambient Topological Classification Problem. *Up to homeomorphism, what are the possible pairs $(\mathbb{R}P^n, \mathbb{R}A)$ where A is a nonsingular real projective algebraic hypersurface of degree m in $\mathbb{R}P^n$?*

These problems are included into Hilbert's sixteenth problem, with an emphasis on the special cases of plane curves and surfaces in $\mathbb{R}P^3$, see [7], [1] and [15].

For $n = 2$ (plane curves) 4.1.A was solved by Harnack [5] for all m, while 4.1.B has been solved only for $m \le 7$. For $n = 3$ (surfaces in $\mathbb{R}P^3$) both 4.1.A and 4.1.B have been solved only for $m \le 4$, for $n \ge 4$ the solutions for 4.1.A and 4.1.B are known only for $m \le 2$.

For $m \le 3$ and $n = 2, 3$ solutions of 4.1.A and 4.1.B coincide, but for $m = 4$ they are different. The simplest example for plane projective curves is a pair of curves of degree 4 each of which consists of two circles, but in one case they are outside each other, in the other one of them envelopes the other one. Both curves can be obtained by a small perturbation of two disjoint circles. (Perturbation is needed, if one wishes to have curves having no singular points even in the complex domain).

For $n = 3$ the simplest example is the following pair of nonsingular surfaces of degree 4 which are homeomorphic to torus. The first of them is defined by equation

$$(x_1^2 + x_2^2 + x_3^2 + 3x_0^2)^2 - 16(x_1^2 + x_2^2)x_0^2 = 0$$

the second is the union of one-sheeted hyperboloid and an imaginary quadric (perturbed, if you wish to have a surface having no singular points even in the complex domain).

The simplest source for restrictions on topology of algebraic surfaces of a given degree is the Bézout theorem. According to it the set of real points of a nonsingular projective hypersurface of degree m and a real line either have at most m common points or the line is contained in the hypersurface, and if the line intersects the set of real points of the hypersurface transversally, then the number of intersection points is congruent to m modulo 2. Topological corollaries of this theorem and its generalizations were extensively discussed in literature since at least, Hilbert's paper [6], see, e.g., in my survey article [16], Theorems 1.3.B-1.3.E and 2.5.A-2.5.D. However, these and other theorems of this kind are better formulated in terms of the regions trees introduced in Section 3 above, see Section 4.4.

4.2. Homology Class Realized by Hypersurface. We start with the most classical topological corollary of the Bézout theorem, for which the region tree is irrelevant.

4.2.A. *The set of real points of a nonsingular hypersurface of degree m is one-sided, if m is odd, and two-sided, if m is even.*

Indeed, by the Bézout theorem a generic line meets a hypersurface of degree m in a number of points congruent to m modulo 2. On the other hand, whether a topological hypersurface in $\mathbb{R}P^3$ is one-sided or two-sided, can be detected by its intersection number modulo 2 with a generic line: a hypersurface is one-sided, iff its intersection number with a generic line is odd. $\square$

4.3. Cubic Hypersurfaces. Here are other well-known restrictions on cubic surfaces.

4.3.A On Number of Cubic's Components. *The set of real points of a nonsingular hypersurface of degree three consists of at most two components.*

Proof. Assume that there are at least three components. Only one of them is one-sided, the other two are of rank 0. Connect with a line the two contractible components. Since they are zero-homologous, the line should intersect each of them with even intersection number. Therefore the total number of intersection points (counted with multiplicities) of the line and the surface is at least four. This contradicts to the Bézout theorem, according to which it should be at most three. □

4.3.B On Two-Component Cubics. *If the set of real points of a nonsingular hypersurface of degree 3 in $\mathbb{R}P^n$ consists of two components, then the components are homeomorphic to the sphere S^{n-1} and projective space $\mathbb{R}P^{n-1}$.*

Proof. Choose a point inside the contractible component. Any line passing through this point intersects the contractible component at least in two points. These points are geometrically distinct, since the line should intersect also the one-sided component. On the other hand, the total number of intersection points is at most three according to the Bézout theorem. Therefore any line passing through the selected point intersects one-sided component exactly in one point and two-sided component exactly in two points. The set of all real lines passing through the point is $\mathbb{R}P^{n-1}$. Drawing a line through the selected point and a real point of the surface defines a one-to-one map of the one-sided component onto $\mathbb{R}P^{n-1}$ and two-to-one map of the two-sided component onto $\mathbb{R}P^{n-1}$. Since the lines are not tangent to the hypersurface, the maps are local diffeomorphisms. Therefore the former is a diffeomorphism and the latter, a two-fold covering. This covering is not trivial, since when one rotates the line by π, the intersection points exchange. Therefore the two-sided component is diffeomorphic to S^{n-1}. □

4.4. Restrictions on Diameters of Trees. Theorem 4.3.A is generalized to the following Theorem. In literature it is formulated in terms of depth of a nest made of components, see, e.g. [16] Theorem 1.3.C.

4.4.A Diameter of Region Tree. *The diameter of the region tree*[1] *of a nonsingular hypersurface of degree m is at most $[m/2]$.*

Proof. Choose two vertices of the region tree the most distant from each other. Choose a point in each of the corresponding regions and connect the points by a line. The obvious image of the line in the region tree is a loop, which passes through the vertices most distant from each other and runs over at most m edges. □

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[1]Here by the diameter of a tree we understand the maximal number of edges in a simple chain of edges of the tree connecting two vertices, i.e., the diameter of the tree with respect to the inner metric such that each edge has length 1.

***4.4.B Extremal Property of* 4.4.A.** (Cf. 4.3.B and [16], 1.3.C.) *If the diameter of the region tree of a nonsingular hypersurface of degree m is $[m/2]$ then the tree is embeddable into a line and the hypersurface consists of $m/2$ spheres enveloping each other, if m is even, or a projective space and $(m-1)/2$ spheres enveloping each other, if m is odd.*

The following theorem seems to be new.

4.4.C 3-Diameter. *Let A be a nonsingular hypersurface of even degree m in $\mathbb{R}P^n$ with $n > 2$. Let $v_1, \dots v_s$ with $s \le 3$ be vertices of the region tree of $\mathbb{R}A$. Assume that v_1, v_2 correspond to regions either with non-trivial linking pairing, or with the sum of ranks equal $n-1$. Then the number of edges in the minimal tree containing $v_1, \dots, v_s$ is at most $m/2$.*

Proof. Choose a point P inside the region corresponding to v_3, if $s = 3$ or just somewhere if $s < 3$. If the sum of the ranks is $n-1$, then by 3.3.B there exists a line passing through P and meeting both L_1 and L_2. If the linking pairing is not trivial, then in each of the regions corresponding v_1 and v_2 take an embedded cycles which have non-zero linking number in $\mathbb{R}P^n$. Denote these circles by L_1 and L_2 respectively. By 2.4.A there exists a line passing through P and meeting both L_1 and L_2.

Thus in either case there exists a line intersecting all the three regions. The corresponding loop in the region tree passes through each edge off the minimal subtree of the region tree containing $v_1, \dots, v_s$ at least twice. On the other hand, by the Bézout theorem it can pass edges at most m times. $\square$

4.4.D Corollary. *Let A be a nonsingular surface of even degree m. Let v_1, v_2, v_3 be vertices of the region tree of $\mathbb{R}A$ and v_1, v_2 correspond to non-contractible regions. Then the number of edges in the minimal tree containing v_1, v_2 and v_3 is at most $m/2$.*

The following Corollary of 4.4.A and 4.4.D is well-known.

4.4.E Corollary. *The set of real points of a nonsingular surface of degree 4 has at most two noncontractible components. If the number of noncontractible components is 2, then there is no other component. If the number of noncontractible components is 1 then the contractible components lie in the same domain outside each other.*

Proof. The case of 3 components such that at least two of them are noncontractible is prohibited by 4.4.C. The case of 1 noncontractible component follows from 4.4.A $\square$

4.4.F *Remark.* In fact, if a nonsingular quartic surface has two noncontractible components then each of them is homeomorphic to torus. It follows from an extremal property of the refined Arnold inequality, see [12]. I do not know, if it can be deduced from the Bézout theorem. However, if one assumes that the domains of the complement which are not adjacent to both components contain lines, then it is not difficult to find homeomorphisms between the components of the surface and the torus, which is the product of these two lines. Cf. the proof of 4.3.B. This seems to be related to Arnold's problem on topology of hyperbolic surfaces in $\mathbb{R}P^3$, see [2].

4.4.G 5-Diameter of Region Tree on Plane. (Cf. [16], 2.5.B) *Let A be a plane nonsingular curve of degree m. Let $v_1, \dots, v_s$ with $s \leq 5$ be vertices of the region tree of $\mathbb{R}A$. Then the number of edges in the minimal tree containing $v_1, \dots, v_s$ is at most m.*

PROOF. Draw a conic through points chosen in the regions corresponding to $v_1, \dots, v_s$. $\square$

4.4.H 5-Diameter of Region Tree in High Dimensions. *Let A be a nonsingular hypersurface of even degree m in $\mathbb{R}P^n$. Let $v_1, \dots, v_s$ with $s \leq 5$ be vertices of the region tree of $\mathbb{R}A$ and at most three of the regions corresponding to $v_1, \dots, v_s$ have rank $< n-2$. Then the number of edges in the minimal tree containing $v_1, \dots, v_s$ is at most m.*

PROOF. Choose points in the regions with rank $< n-2$ and draw a plane through these points. This plane intersects the regions corresponding to the rest of $v_1, \dots, v_s$, for they have rank $n-2$. Choose points in the intersection of the plane with these regions and draw a conic through the 5 chosen points. $\square$

Notice that for dimension ≥ 4 there is at most one region of rank $\geq n-2$. Hence in 4.4.H the number s can be at most 4. For $n = 3$ the number of regions under consideration can be 5, and in this case two of regions should be noncontractible.

It seems to be impossible to formulate in a compact way all the corollaries of the Bézout theorem of this sort. Bézout theorem belongs to Algebraic Geometry, while the corollaries are formulated topologically. The transition necessarily involves a loss of the contents. Region trees provide terms just slightly more adequate for this transition than the terms used before. A systematic intervention of a real version of the Schubert calculus may give rise to further more sophisticated formulations for higher-dimensional cases.

5. Problem on Number of Noncontractible Components

5.1. Harnack's Inequality and Problem. Surprisingly, the Bézout theorem gave much lesser restrictions in the case of surfaces than in the case of plane curves.

Recall, that the Harnack inequality [5] which was the very first result of general kind on the topology of real plane projective algebraic curves of degree m was deduced from the Bézout's theorem. The Harnack inequality says that the set of real points of a nonsingular plane projective curve of degree m has at most $\frac{1}{2}(m-1)(m-2)+1$ connected components. Harnack, in the paper [5], where he discovered the Harnack inequality, asked if there exists a similar inequality for surfaces in the three-dimensional projective space. This question is known as the Harnack problem.

For the case of cubic surfaces the maximal number of components is 2 and this is proved using the Bézout theorem. Generalization of this approach to the higher degrees gives an estimate of diameter of regions tree 4.4.A, but not an estimate of the number of connected components. For quartic surfaces the number is 10. This was proved by Kharlamov [10] by methods much more sophisticated than the Bézout's theorem. The answer for higher degrees is unknown. The best known polynomial estimate is $\frac{5}{12}m^3 - \frac{3}{2}m^2 + \frac{25}{12}m$. It is obtained as a sum of the Smith-Thom inequality

$$dim_{\mathbb{Z}_2} H_*(\mathbb{R}A; \mathbb{Z}_2) \leq dim_{\mathbb{Z}_2} H_(\mathbb{C}A; \mathbb{Z}_2) = m^3 - 4m^2 + 6m$$

and the Comessatti-Petrovsky-Oleinik inequality

$$\chi(\mathbb{R}A) \leq h^{1,1}(\mathbb{C}A) = \frac{2}{3}m^3 - 2m^2 + \frac{7}{3}m.$$

For $m = 5$ it gives 25, while the best known example consists of 22 components [9].

5.2. Number of Noncontractible Components. At first glance the question on the maximal number of *noncontractible* components of the set of real points of a real projective surface of degree m is similar to the Harnack problem. However, this number is estimated by a quadratic function of m.

5.2.A. *The number of noncontractible connected components of the set of real points of a real projective surface of degree m is not greater than* $\frac{1}{2}(m-1)(m-2)+1$.

PROOF. Take any plane transversal to the surface. Each noncontractible component intersects it. Therefore the number of components of the intersection is not less the number of noncontractible components of the surface. On the other hand, the intersection is the set of real points of a nonsingular plane projective curve off degree m and by the Harnack inequality the number of its components is at most $\frac{1}{2}(m-1)(m-2)+1$. $\square$

5.2.B *Remark.* Of course, Theorem 5.2.A is not trivial only for even m: a surface of odd degree has only one noncontractible component.

It would be interesting also to estimate the number of components of rank r of a hypersurface of even degree m in $\mathbb{R}P^n$. The sharp estimate is known for $r > n/2$: it is one, see Section 3.2.

5.3. Surfaces of Even Degree with Many Noncontractible Components. For $m = 4$ the estimation of Theorem 5.2.A is not sharp: it gives 4, while by Theorem 4.4.E the number of noncontractible components is at most 2.

For the higher degrees the question remains open. The best examples that I know have about twice less noncontractible components than Theorem 5.2.A suggests. More precisely, $\frac{1}{4}(m-2)^2+1$ noncontractible components. These surfaces can be constructed by a version of classical construction basically due to Harnack [5] as follows.

Let us start from a one-sheeted hyperboloid A. This is the first surface in the series under construction. Let it be defined by a polynomial H.

To construct the next one, take a real curve C_1 on A, which can be cut on A by a surface of degree 2 and has the set of real points consisting of two disjoint branches isotopic on $\mathbb{R}A$ to its generatrix. This curve can be obtained as the union of two real generatrices of one family and two imaginary conjugate generatrices of the other one. Let F_1 a real homogeneous polynomial of degree 2 in 4 variables which defines a surface cutting A in C_1. Then the polynomial $H + \varepsilon_1 F_1$ with sufficiently small $\varepsilon_1 > 0$ defines a hyperboloid which is close to A and intersects it in C_1.

Take the union of the original hyperboloid and the new one. It is defined by $H(H + \varepsilon_1 F_1)$. The second surface of our series is obtained from this union by a small perturbation.

Take a real curve C_2 on A. It has to be cut on A by a surface of degree 4 and have the set of real points consisting of 4 branches isotopic on $\mathbb{R}A$ to a generatrix and contained in a single component of $\mathbb{R}A \smallsetminus \mathbb{R}C_1$. This curve can be obtained as the union of 4 real generatrices of one family and 2 pairs of imaginary conjugate generatrices of the other one. Let F_2 is a real homogeneous polynomial of degree 4 in 4 variables, which defines a surface cutting A in C_2. The surface A_1 defined by polynomial

$$H(H + \varepsilon_1 F_1) + \varepsilon_2 F_2$$

with sufficiently small $\varepsilon_2 > 0$ is the second surface in the series under construction. It is easy to see that its real part is isotopic to disjoint union of 2 one-sheeted hyperboloids.

Take the union of the original hyperboloid A and A_1. It is defined by

$$H(H(H + \varepsilon_1 F_1) + \varepsilon_2 F_2).$$

The third surface of our series is obtained from this union by a small perturbation.

To make the perturbation, let us take a real curve C_3 on A. It has to be cut on A by a surface of degree 6 and have the set of real points consisting of 6 branches isotopic on $\mathbb{R}A$ to a generatrix and contained in a single component of $\mathbb{R}A \smallsetminus \mathbb{R}C_2$. This curve can be obtained as the union of 6 real generatrices of one family and 3 pairs of imaginary conjugate generatrices of the other one. Let F_3 is a real homogeneous polynomial of degree 6 in 4 variables, which defines a surface cutting A in C_3. The surface A_2 defined by polynomial

$$H(H(H + \varepsilon_1 F_1) + \varepsilon_2 F_2) + \varepsilon_3 F_3$$

with sufficiently small $\varepsilon_3 > 0$ is the third surface in the series under construction. It is easy to see that its real part is isotopic to disjoint union of 5 one-sheeted hyperboloids.

Then the construction continues in the same way. The set of real points of a surface of degree m constructed in this way consists of $\frac{1}{4}(m-2)^2+1$ noncontractible components homeomorphic to torus.

This construction can be modified to change topological types of components and add some contractible components. However, I do not see any possibility to increase the number of noncontractible components. I would rather expect an improvement of Theorem 5.2.A.

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