Алгебра и анализ
Том 1 (1989), вып. 4

Leningrad Math. J.
Vol. 1 (1990), No. 4

Light Reading for the Professional

This article initiates a new section of the journal, devoted to what can be called "informal surveys". This genre, which is new for us, is well known and increasingly popular in other countries—one need only recall the entertaining mathematical (and mathematics-related) "stories in pictures" in the Mathematical Intelligencer, American Mathematical Monthly, *and elsewhere. The editors hope that you, the reader, will try your hand at this difficult genre, in order to facilitate greater understanding between mathematicians, who are (alas!) divided into narrow professional specialties.*

Articles for this section will be treated as surveys (albeit informal ones), i.e., they must be agreed upon with the editors at the planning stage.

The editors would like to thank S. G. Gindikin for proposing the idea of this section.

CONFIGURATIONS OF SKEW LINES

O. YA. VIRO AND YU. V. DROBOTUKHINA

ABSTRACT. This article is a survey of recent results on projective configurations of subspaces in general position. It is written in the form of a popular introduction to the subject, with much of the material accessible to advanced high school students. However, in the part of the survey concerning configurations of lines in general position in three-dimensional space we give a complete exposition.

The only new material here is the construction of a suspension for configurations of subspaces in projective space which increases the dimension of the ambient space by 4 and the dimension of the subspaces by 2 and preserves a large part of the algebro-topological characteristics of the configuration (see the penultimate page of the article)

We recently wrote an article for the journal "Kvant" in which we described some recent research of a completely elementary nature. The article appeared in the journal in the third issue of 1988, in a shortened form. But here we would like to expand the article in order to encompass or at least mention some related questions. Thus, the present paper is an expanded version of the article that appeared in "Kvant". It may be regarded as a survey of recent results in the topology of configurations in general position. We decided to keep the style of an article for "Kvant", in the hope that it would also be appreciated by the professional mathematician. If the reader finds the style irritating, we apologize, and we mention that the material in the first two-thirds of the article (through the section on "sets of five lines") is announced in the note [1], and the final third of the article is written in a more traditional style.

The article in "Kvant" was titled *Interlacing of skew lines.* This title sounds a little strange, doesn't it? The word "interlacing" suggests something flexible, not straight lines! To be sure, the title refers not to be the process of interlacing, but rather to the result. But is it possible to weave together skew lines which are situated in some clever way with respect to one another? At first glance this

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 51H10, 51A20; Secondary 05B30, 51E30.

Key words and phrases. Projective configurations, skew lines, isotopies and rigid isotopies, mirror configurations, join.

may seem not to be possible. Yet where do we get this impression? In daily life we never come across anything that really resembles a straight line. What bothers us is not that there is not such thing as an infinitely thin object—we are prepared to neglect the thickness—but rather that there is no such thing as an infinitely long object. Even light rays—which are models of linearity—become scattered and dispersed, and cannot be detected at a large distance. In practice one deals only with line segments.

Any set of disjoint line segments can be moved around to any other relative location in such a way that they remain disjoint. This we can see from experience, and it is also not hard to prove. We depict straight lines using line segments, and so it seems to us that straight lines cannot be woven together. But is that really the case?

First of all, let us give a more precise statement of the questions which concern us. The first question is: Can a set of disjoint lines be rearranged? But what do we mean by the term "rearrange"? Here we shall not be concerned with the angles or distances between the lines. We shall consider the relative position of the lines to be unchanged if we move them in such a way that they never touch. But if one set of lines cannot be obtained from another set by such a movement, then we shall say that the two sets of lines are arranged differently.

The simplest lines for us to visualize are parallel lines. Clearly, any two sets of parallel lines with the same number of lines in each set have the same arrangement. In fact, if we consider the lines of one set to be "frozen" in place and then rotate the entire space, we can make them parallel to the lines of the other set; then, moving the lines of the first set one by one in such a way that they remain parallel and do not bump into one another, we can easily make them coincide with the lines of the second set.

We now consider arbitrary sets of lines. Can an arbitrary set of lines be moved ("combed") into a set of parallel lines? This question has a simple and unexpected answer, which is hard to arrive at by considering concrete sets of lines. If you take a specific set of lines and study it for a while, you can probably find a way to make all of the lines parallel. But this does not give an answer to the question in full generality, because you undoubtedly made use of some specific features of your set of lines. Can one treat all possible sets of lines at once? It turns out that one can, and this is how. Let us take an arbitrary set of disjoint lines. We choose two parallel planes which are not parallel to any of the lines in our set. We fix the points of intersection of the first plane with the lines, fastening the lines at those points. We also fix the intersection of the lines with the second plane, but only as a point on that plane, which we allow to slide along the lines. In other words, we drill small holes in the second plane where it intersects with the lines. We then move the second plane away from the first one in the direction perpendicular to both planes. The lines are pulled through the little holes, and the angles which they form with the planes increase. If we move the second plane to infinity in a finite amount of time, then these angles all reach 90°, i.e., the lines become parallel to one another. This "combing" of our set of lines can be described as follows in a language which is more customary for geometry: we expand the space away from the first plane in a direction perpendicular to it, where the expansion factor increases rapidly to infinity in a finite length of time. Here the straight lines rotate around their points of intersection with the plane, and in the limit they become perpendicular to the plane.

Thus, one cannot have interlaced disjoint lines: all sets of disjoint lines have the same arrangement. But our title refers to skew lines, and so sets of parallel lines are excluded. There is a serious reason for this. Parallel lines are very close to being intersecting lines: one can move one of two parallel lines by an arbitrarily small amount so as to make them intersect. This is not the case for skew lines.

Since we have decided not to allow parallel lines, we must reexamine the question of which sets of lines have the same arrangement and which do not. We shall say that the arrangement of a set of lines remains the same if it is moved in such a way that the lines are always skew, never parallel. In what follows we will often be considering such movements of lines, and so it is useful to have a special word to refer to them. We shall use the word *isotopy* to denote such a movement of lines. If one set of lines cannot be obtained from another by means of an isotopy, then we say that the two sets have different arrangements. We shall also say that such sets of lines are nonisotopic.

The amount of difficulty in determining whether two sets of lines are isotopic depends most of all on the number of lines in the sets. In general, the more lines, the more clever one must be to find an isotopy which transforms one set into the other. We first treat the simplest case of the isotopy problem.

Two lines

We take any two pairs of skew lines, and try to decide whether they are isotopic. In this case it is perhaps too pretentious to use the word "problem", because it is completely obvious that we have an isotopy. Nevertheless, we shall make a detailed examination of the proof.

Using a rotation around a line which is perpendicular to both lines in one of the pairs, we can make the angle between the lines the same in both pairs; in fact, we can make both angles 90°. We note that the smallest line segment joining the two lines in a pair is the segment of the common perpendicular which is contained between them. We next bring the two lines closer together (or move them farther apart) along this perpendicular, so that the segments have the same length for the two pairs; after that we move one pair so that the segment between the two lines coincides with the segment for the other pair. We use a rotation around this segment to make one of the lines of the first pair coincide with a line of the second pair (this can be done because all of the lines are perpendicular to the segment). In the process the second lines of the pairs also come together. In fact, they both pass through a common point—an endpoint of the perpendicular segment—and are perpendicular to the same plane—the plane determined by the perpendicular and the first lines of the pairs (which now coincide). The proof is complete.

At the end of the proof, after we made the distances between the two lines the same for the two pairs, we moved a pair of lines in a rigid manner—without changing either the distance or the angle between them. The question arises: Suppose that both the distances and angles between the two lines are the same for two pairs of skew lines. Is it always possible to find an isotopy between the two pairs during which the distance and angle remain fixed? The previous argument shows that this question has an affirmative answer if the angle is 90°. However, if the angle is not 90°, then it may happen that after the isotopy in the previous paragraph the second lines in the pairs do not coincide. This unlucky

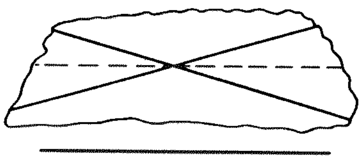


FIGURE 1

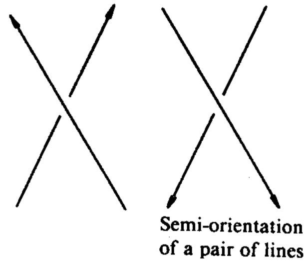


FIGURE 2

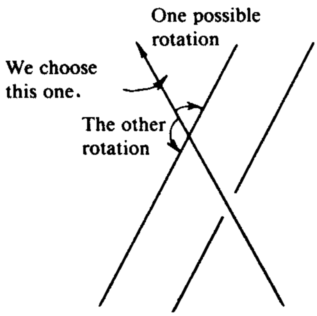


FIGURE 3

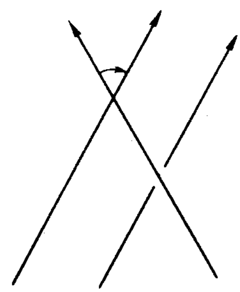


FIGURE 4

case is illustrated in Figure 1. The second lines in the pairs form an angle whose bisector is parallel to the first (skew) lines, and the plane containing the second lines is perpendicular to the plane containing the bisector and the first lines. Thus, there was a good reason why we wanted to make the angles 90° in the beginning of the above proof: for any other choice of the angle, the construction would not give the desired result. But this was not simply an artifact of our particular construction; it turns out that any two pairs of skew lines with equal distance and angle which do not coincide after the above construction cannot be made to coincide using any isotopy during which the distance and angle remain fixed. This is connected with a remarkable phenomenon, which we shall encounter often in the sequel. It merits a more detailed discussion.

Orientations and semi-orientations

To orient a set of lines means to give a direction to each line in the set. There are 2^n possible orientations of a set of n lines. A *semi-orientation* of a set of lines is a pair of opposite orientations (Figure 2).

Any pair of nonperpendicular lines has a canonical semi-orientation which is determined by the relative position of the two lines. Namely, we choose an arbitrary orientation of one of the lines, and then we determine the orientation of the second line by rotating the first line in the most economical way (i.e., with the smallest angle of rotation) so as to make it parallel to the second line (see Figure 3). We then give the second line the orientation pointing in the same direction as the (now parallel) first line (Figure 4). Thus, choosing an orientation of one of the lines determines an orientation of the pair. If we choose the opposite orientation of the first line, then we obtain the opposite orientation of the pair. If we were to use the other line to start with, we would obtain the same pair of opposite orientations. These two opposite orientations are what we meant by the canonical semi-orientation of the pair of nonperpendicular lines.

An isotropy during which the angle between the lines remains fixed takes the canonical semi-orientation to the canonical semi-orientation. This suggests the idea of considering another type of isotopy—isotopies of semi-oriented pairs of skew lines. Here we allow the angle and distance between the lines to change, but we require that the semi-orientation be preserved. Such an isotopy occupies an intermediate position between an arbitrary isotopy and an isotopy during which the distance and angle (where we suppose that the angle is $\neq 90^\circ$) remain fixed. That is, if there is no semi-oriented isotopy between two semi-oriented pairs of lines, then there is certainly no isotopy between them which preserves the distance and angle. What can stand in the way of an isotopy of semi-oriented pairs of lines?

The linking coefficient

Any semi-oriented pair of lines has a characteristic which takes the value $+1$ or -1. It is called the *linking coefficient.* This coefficient is preserved under isotopies, and so if two semi-oriented pairs of lines have different linking coefficients, then they are not isotopic. Here is the definition of the linking coefficient. The most economical way of aligning an oriented line with a second oriented line which is skew to it is to place it alongside a common perpendicular to the two lines and then rotate it by the smallest angle that brings it to the same direction as the second line. Here the line rotates either like the right hand around the thumb, or like the left hand (Figure 5, next page). In the first case the linking coefficient is -1, and in the second case it is $+1$.

To help the reader familiar with algebraic topology make the right connection, we give a second equivalent definition of the linking coefficient of a pair of oriented skew lines. Through one of the lines we draw a plane which intersects the other line. We place our right hand so that our thumb rests on the second line and passes through the plane in the direction determined by the orientation of the line, while rotating in the direction our fingers point. On the plane we obtain an oriented circle which is traced by the tips of our fingers. The orientation of the circle may be the same as the orientation of the first line (Figure 6) or different (Figure 7). In the first case the linking coefficient is $+1$, and in the second case it is -1. Figure 8 will enable the reader to see that the two definitions of the linking coefficient are equivalent.

It is clear that changing the orientation of one of the lines of the pair changes the linking coefficient. Hence, if the orientation of the pair is changed to the opposite orientation (i.e., the orientation is reversed on both lines), then the linking coefficient does not change. In other words, the linking coefficient is an invariant of a semi-oriented pair: it depends only on the semi-orientation. If we look at the reflection of our pair of oriented lines in the mirror (Figure 9), the linking coefficient changes.

We now return to the unfortunate situation we encountered when looking for an isotopy between two pairs of skew lines which preserves the distance and angle between the lines (see Figure 10). At the time we could not answer the question of whether the sets are isotopic (Figure 11). Now, however, we see that these pairs (with their canonical semi-orientation) are obtained from one another by a mirror reflection, and so they have different linking coefficients. Thus, they cannot be connected by an isotopy which preserves the distance and angle between the lines. But if two pairs have the same distance and angle and also the same linking coefficient, then they can be connected by such an isotopy.

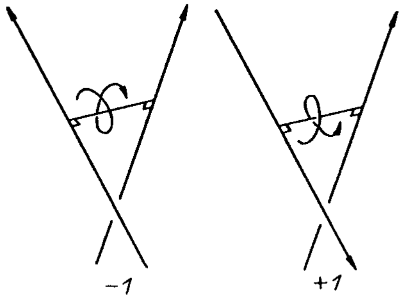


FIGURE 5

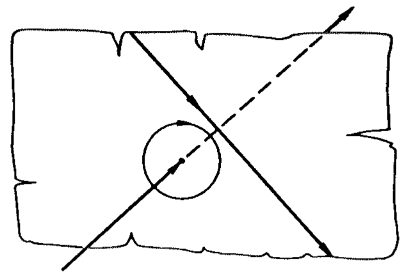
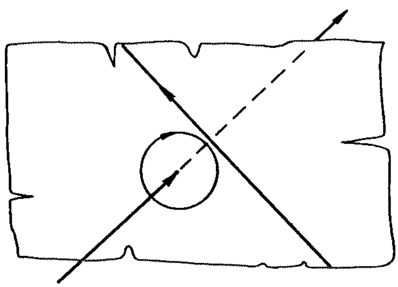


FIGURE 6



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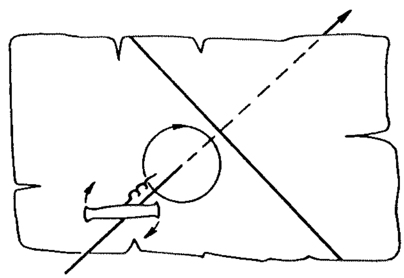

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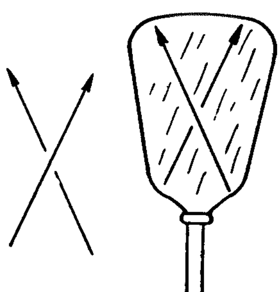


FIGURE 9

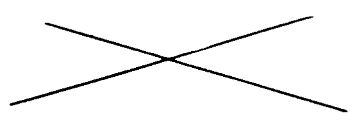


FIGURE 10

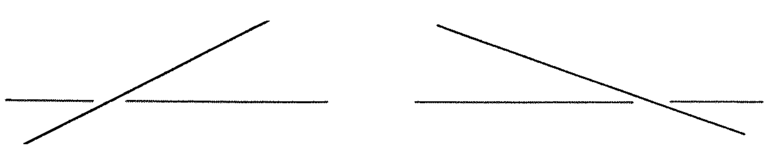


FIGURE 11

By the way, it is possible to modify the notion of the angle between two skew lines in such a way as to incorporate the linking coefficient and thereby make it unnecessary to work with the linking coefficient separately. The angle between two lines was defined above so as to be in the interval $(0^\circ, 90^\circ)$. We define the modified angle between two lines to be the product of the angle in the earlier sense and the linking coefficient, if the latter is defined (i.e., if the angle is not 90°), and to be the angle in the earlier sense (i.e., 90°) if the linking coefficient is not defined. The modified angle is in one of the intervals $(-90^\circ, 0^\circ)$, $(0^\circ, +90^\circ]$. The sign can be determined from the right hand rule, without saying anything about the linking coefficient.

We have thereby completely analyzed the situation with sets of two skew lines.

Triples of lines

When we studied pairs of lines, an important role was played by the common perpendicular to the two skew lines. Strictly speaking, we could have avoided using it; but it seemed to be connected to the lines in such a natural way, providing a tangible tie between them, that it would have been strange not to make use of it. Now it would be good to find something equally natural for a triple of skew lines. There are two objects that are capable of playing this role. We shall discuss one of them now, and postpone consideration of the second one. Jumping ahead, suffice it to say that the second object is a hyperboloid.

These objects will not be associated to every triple of pairwise skew lines. We will have to disallow triples whose lines lie in three parallel planes. But notice that such an arrangement is unstable: by nudging one of the lines a little, we obtain an isotopic triple to which our constructions can be applied.

Thus, we consider an arbitrary triple of pairwise skew lines which do not lie in three parallel planes. For each line we draw two planes containing the line, each parallel to one of the other two lines. In this way we obtain six planes, i.e., three pairs of parallel planes. These planes intersect to form a parallelepiped. Our lines are the extensions of three of its skew edges (Figure 12). Thus, any three pair-wise skew lines which do not lie in three parallel planes are extensions of the edges of a certain parallelepiped. This parallelepiped is the first object which we associate to the triple of lines. What is special about it? In the first place, it is unique. In fact, there is a unique plane parallel to a given line that contains a second skew line; and if these lines are the extensions of edges of a parallelepiped, then this plane contains one of its faces. Consequently, the six planes are uniquely determined by the original triple of lines; since any parallelepiped whose edges lie on these lines is bounded by those planes, it is also uniquely determined.

We see that the parallelepiped joins together the lines of the triple just as nicely as the common perpendicular joined together the lines of the pair. Just as in the case of the common perpendicular and the semi-oriented pair of non-perpendicular lines, the original geometry of the configuration naturally leads to

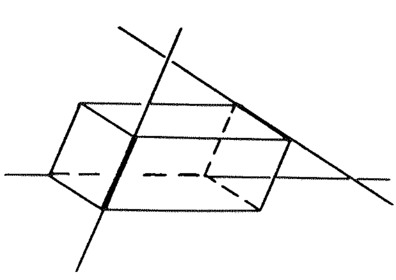


FIGURE 12

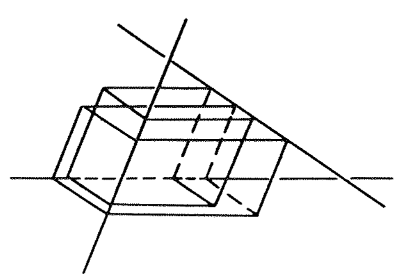


FIGURE 13

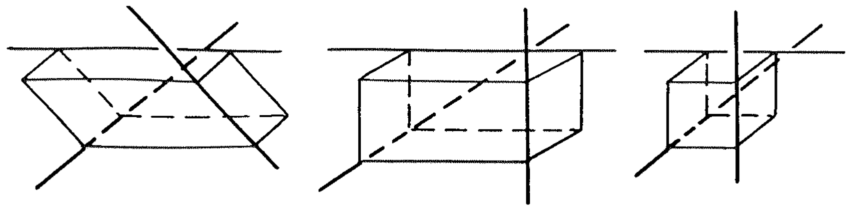


FIGURE 14

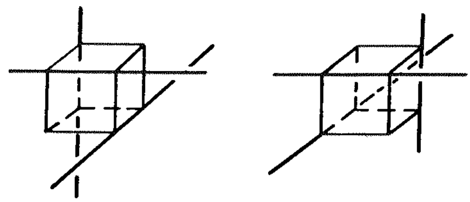


FIGURE 15

something more, though still something which is connected with it in a canonical way and so merits our further consideration when we study the original object.

A RIDDLE. In Figure 13, despite what was proven above, we have drawn two different parallelepipeds with edges lying on three pairwise skew lines. What is going on?

We now look at the classification of triples up to isotopy. As shown above, we may suppose that the lines in the triple are extensions of edges of a certain parallelepiped. A parallelepiped is determined (up to translation) by the lengths of its edges and the angles between them. Using a continuous deformation, we can first make all of the angles into right angles (obtaining a rectangular parallelepiped), and then we can make all of the edges have the same length, for example, length one (obtaining a cube) (Figure 14). This deformation induces an isotopy of the triple of lines which are extensions of edges of the parallelepiped. In this way we have managed to place the lines of our triple along pairwise skew edges of a unit cube. This is a remarkable accomplishment. It means that we now know that there are not very many possible nonisotopic sets of three skew lines—there are at most the number of triples of skew edges on a cube, and this number is 8. And even 8 is too many. We can use a rotation of the cube to take any edge of the cube to any other edge, and this reduces the number of possible nonisotopic configuration types to two. They are shown in Figure 15.

This success might prompt us to hope that we can similarly find an isotopy between the two triples of lines in Figure 15, and thereby prove that all triples of skew lines are isotopic. Try to do this!

You're having trouble? Don't blame yourself—it cannot be done! Just like pairs of oriented lines, triples of (nonoriented) lines have an invariant, also called the *linking coefficient*, which takes the value $+1$ or -1, is preserved under isotopies, and changes when one takes a mirror reflection of the triple of lines. Here is its definition. Suppose we have a set of three pairwise skew lines. We orient the tree lines in an arbitrary way. Then each pair of lines in the triple has a linking coefficient (equal to ± 1). If we multiply all of the linking coefficients, we obtain a number (also ± 1), which is what we call the linking coefficient of the original triple of lines. This coefficient does not depend on the orientation of the lines, since if we reverse the orientation of any line, the effect is to change the linking coefficients of two of the pairs, and this does not change the product. The fact that the linking coefficient of a triple is preserved under isotopy and changes under mirror reflection follows from the corresponding properties of the linking coefficients of pairs of oriented lines. Since the triples of lines in Figure 15 are the mirror images of one another, they have different linking coefficients, and hence they are not isotopic to one another.

Since any triple of pairwise skew lines is isotopic to one of the two triples in Figure 15, it follows that two triples of lines are isotopic if and only if they have the same linking coefficient.

Thus, as soon as we reach three lines we find that there are different possible arrangements of triples of skew lines. This provides a justification for the title of the paper, and for our subsequent use of the word *interlacing* for a set of pairwise skew lines.

PROBLEM. It is natural to expect that the linking coefficient of a triple of nonoriented lines is equal to the linking coefficient of some pair of semi-oriented lines which can be constructed from the triple in a canonical way. This is in fact the case, except that rather than one such semi-oriented pair there are three. Prove that for any triple of skew lines there is a unique semi-orientation such that the linking coefficients of all three pairs of lines in the triple are equal. Obviously, this value is also equal to the linking coefficient of the triple.

Mirror and nonmirror sets

We note that a triple of skew lines is never isotopic to its mirror image, while a pair of lines is isotopic to its mirror image. In general, we say that a set of pairwise skew lines has the *mirror* property if it is isotopic to its mirror image; otherwise we say it is a *nonmirror* set. Thus, a triple is always a nonmirror set, and a pair is a mirror set. The following questions arise:

1) Are there other values of p such that any interlacing of p lines is nonmirror?

2) Are there other values of p such that any interlacing of p lines is mirror?

3) For what p do there exist nonmirror interlacings of p lines?

4) For what p do there exist mirror interlacings of p lines?

Although this does not take us very far in the direction of an answer to our original question (of describing the set of interlacings of p lines up to isotopy), it is worthwhile to take up these four questions. They are rough and somewhat superficial questions, but at the same time they have a more qualitative character. Because of this roughness and superficiality we can be confident of early success, and the result will undoubtedly be useful in our classification.

We do not yet have at our disposal very many tools for proving the nonmirror property. But we do know that every triple is nonmirror, and this is already a lot. After all, any set of more than three lines contains triples. Each triple changes its linking coefficient in the course of a mirror reflection. Thus, if the interlacing has the mirror property, then it must have the same number of triples with linking coefficient $+1$ as with linking coefficient -1. In particular, the total number of triples in the interlacing must be even. This simple argument leads us to the following unexpected result.

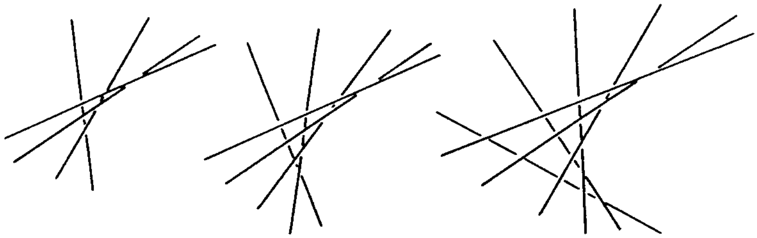


FIGURE 16

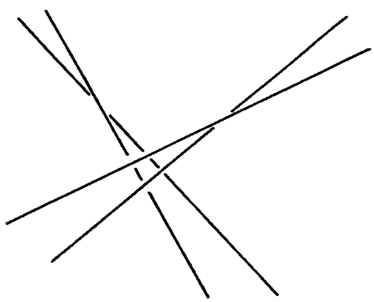


FIGURE 17

THEOREM 1. *If $p \equiv 3 \bmod 4$, then every interlacing of p lines is nonmirror.*

PROOF. The number of triples in an interlacing of p lines is equal to $p(p-1)(p-2)/6$, and this is odd if and only if $p \equiv 3 \bmod 4$. ●

Theorem 1 gives an affirmative answer to the first of the four questions above. The second question has a negative answer: for any $p \geq 3$ one can construct a nonmirror interlacing of p lines. This also answers question 3). The simplest nonmirror interlacings are shown in Figure 16 for $p = 4, 5$, and 6. It is easy to continue with this sequence of examples. All of the triples of lines in the interlacings in this sequence have the same linking coefficient, and for this reason we know that the interlacings are nonmirror.

It remains to answer Question 4). We do not yet know whether or not there ae mirror interlacings of p lines when $p \not\equiv 3 \bmod 4$. It is convenient to consider separately the two cases: p even, and $p \equiv 1 \bmod 4$, although in both cases the question turns out to have a positive answer. In Figure 17 (in which $p = 4$) we show the simplest example of a mirror interlacing of p lines with p even. For any even number p, we take two sets of $p/2$ lines, one behind the other. The lines of the set nearest us are taken from the sequence of nonmirror interlacings constructed above (see Figure 16). The other set of $p/2$ lines is obtained from the first by rotating and then reflecting in a mirror. How do we see that the interlacing in Figure 17 has the mirror property? We move the set that is nearest us in such a way that the part of its projection which contains all of the intersections (in the projection) passes over and above the projection of the other set (Figure 18). If we then rotate Figure 18 by 90° clockwise, we see that we obtain the mirror image of the original interlacing.

We now turn to the case $p \equiv 1 \bmod 4$, i.e., $p = 4k + 1$. A mirror interlacing with $k = 1$ is shown in Figure 19. Four of the lines form two pairs which are situated as in the mirror interlacing of four lines constructed above. The fifth

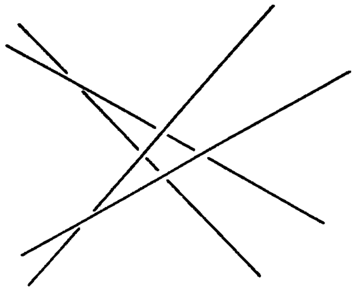


FIGURE 18

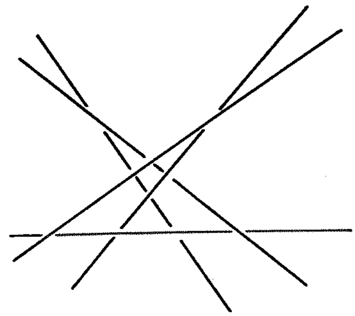


FIGURE 19

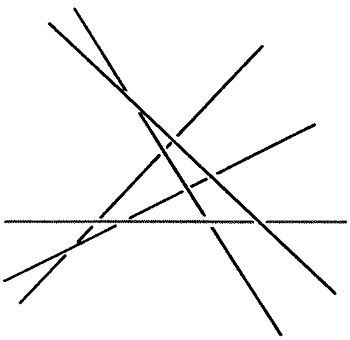


FIGURE 20

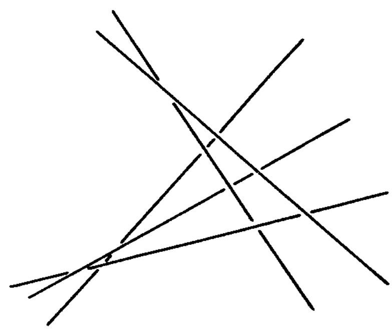


FIGURE 21

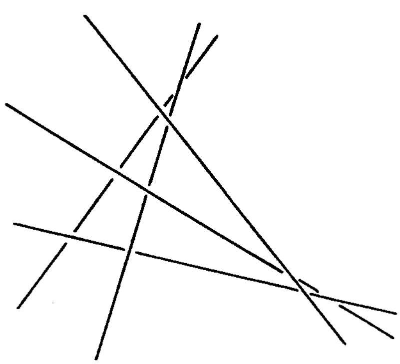


FIGURE 22

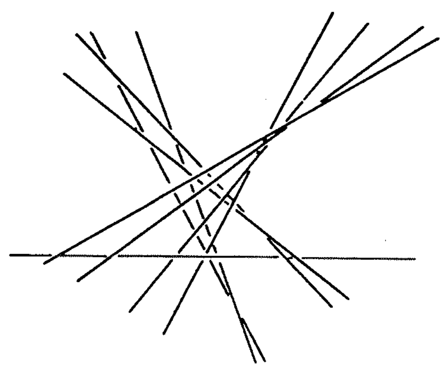


FIGURE 23

line is placed so as to separate the two lines in each pair. An isotopy between this interlacing and its mirror image can be constructed as follows. We rotate the lines of the pair nearest us around the fifth line by almost 180°—until the lines of the other pair are in the way (Figure 20). We then move the fifth line so that its projection passes to the other side of the intersection (in the projection) of the lines that we moved before (Figure 21). It remains simply to look at the resulting interlacing from the opposite side. We do this by rotating it by 180° around a vertical line (Figure 22). Now we see that we have the mirror image of the original interlacing.

Using this example, it is easy to manufacture mirror interlacings of $4k+1$ lines for $k>1$. Each line in Figure 19 except for the fifth is replaced by an interlacing of k lines which either is taken from the sequence in Figure 16 or else is the mirror reflection of an interlacing in that sequence. This must be done in such a way that the interlacings which replace the lines of one of the pairs form an interlacing of the same type. See Figure 23 for the case $k=2$. There is no work needed to prove that the final result is a mirror interlacing, since the required isotopy can be obtained in the obvious way from the one in the previous paragraph.

Four lines

At this point we have actually already encountered all of the types of interlacings of four lines. There are three of them, and they are depicted in Figure 24. The interlacing in Figure 16 is on the left, its mirror image is in the center, and the interlacing in Figure 17 is on the right. We have already proved that these three sets are not isotopic to one another: the first one is not a mirror interlacing, and so is not isotopic to the second one, and the third one is a mirror interlacing, and so is not isotopic to either the first or the second.

In order to show that any interlacing of four lines is isotopic to one of the interlacings in Figure 24, we shall have to make use of the second of the two objects which, as mentioned above, are associated to a triple of lines. This is a one-sheeted hyperboloid—a surface which is usually studied in analytic geometry. There one learns that a one-sheeted hyperboloid (henceforth referred to simply as a hyperboloid) is made up of lines—its generatrices. Any two generatrices in the same family are skew, while any two generatrices in different families are either parallel or intersect. We list some other properties of hyperboloids which we shall need:

(1) if a line has three points in common with a hyperboloid, then it is a generatrix;

(2) a plane containing a generatrix of a hyperboloid intersects the hyperboloid in two generatrices;

(3) there is a hyperboloid passing through any three pairwise skew lines which do not lie in parallel planes.

These properties are simple consequences of the fact that a hyperboloid is a surface of degree two. Of course, one could describe all of this without appealing to analytic geometry, using the same language as the ancient Greeks, but we shall not try the reader's patience by proceeding in that way.

Thus, in order to complete the isotopy classification of four-tuples of lines, we shall prove that any interlacing of four lines is isotopic to one of the interlacings in Figure 24. We take an arbitrary interlacing of four lines. By moving it slightly, if necessary, we can obtain a situation where three of the four lines (it makes no difference which three) do not line in parallel planes. We construct a hyperboloid through these three lines, and we observe how the fourth line is situated relative to the hyperboloid. There are four possibilities:

(a) the line does not intersect the hyperboloid;

(b) the line intersects the hyperboloid in a single point;

(c) the line intersects the hyperboloid in two points;

(d) the line lies on the hyperboloid.

In case (d) the interlacing of four lines consists of four generatrices of the hyperboloid, and is obviously isotopic to the left or the center interlacing in Figure 24.

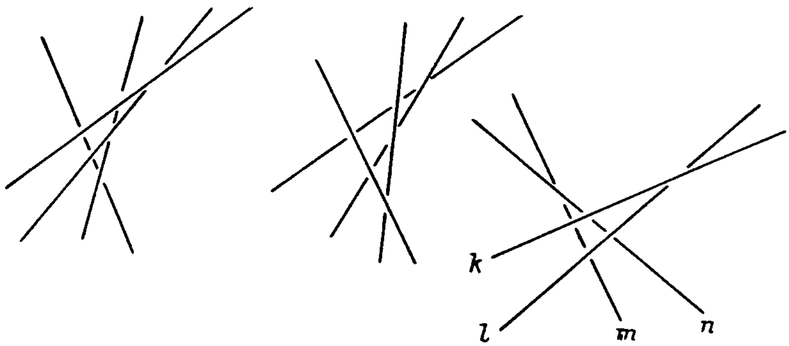


FIGURE 24

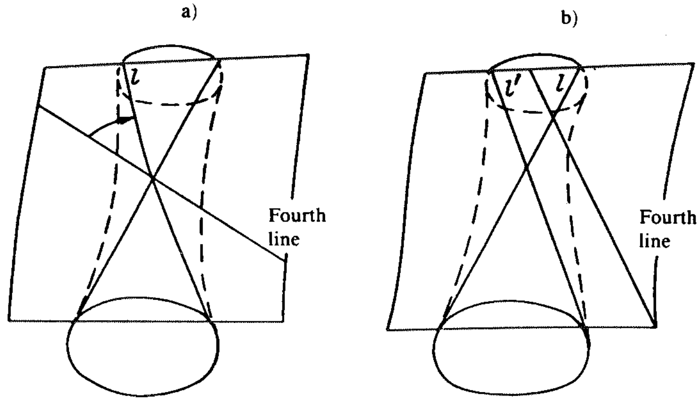


FIGURE 25

If the fourth line does not intersect the hyperboloid, then it can be brought in toward the hyperboloid until it is tangent to the hyperboloid, i.e., the first case can easily be reduced to case (b).

Case (b), in turn, reduces to either (c) or (d). To see this, we draw a generatrix l through the point of intersection of the hyperboloid with the fourth line, where l is taken in the same family of generatrices as the first three lines of the interlacing. By property (3), the plane a containing l and the fourth line intersects the hyperboloid in two generatrices l and l'. If l' intersects l and the fourth line in the same point, then, rotating the fourth line around this point of intersection in the plane a until it coincides with l, we find ourselves in case (d) (Figure 25(a)). Otherwise, the fourth line of the interlacing is parallel to l' (if this weren't the case we would have case (c)) (see Figure 25(b)). But if we perform a small rotation toward the fourth line around the intersection point with l in the plane a, we see that the fourth line is no longer parallel to l': it intersects l', and hence it intersects the hyperboloid in two points, giving us case (c).

Now if the fourth line intersects the hyperboloid in two pints, then everything depends on whether these points are in the same part of the hyperboloid into which the first three lines divide it, or are in different parts (the hyperboloid is divided into three sections). If they are in the same part, then the fourth line can be placed on the hyperboloid without the first three lines interfering. Then the fourth line becomes a generatrix, and we are in case (d). If the fourth line intersects the hyperboloid in different parts, then the interlacing is isotopic to the right interlacing in Figure 24.

Isotopic lines of an interlacing

The next step—the classification of interlacings of five lines—requires a more careful study of the inner structure of interlacings. The reader has undoubtedly noticed the striking difference between mirror and nonmirror interlacings—compare the sets of lines in Figure 24. The left and the center interlacings both have the feature that any line of the interlacing can be taken to any other line by means of an isotopy. This is not the case for a mirror interlacing (the right one in Figure 24). We shall say that two lines of an interlacing are *isotopic* if there exists an isotopy of the interlacing which takes one of the lines to the other one. To be sure, strictly speaking this is not an isotopy, because at the last moment the two lines come together. Instead of changing the meaning of the word "isotopy", we are better off leaving the meaning unchanged and adopting the following definition of isotopic lines of an interlacing: there is an isotopy of the entire interlacing which makes the two lines approach one another so that one can separate them from the other lines of the interlacing by a hyperboloid (in which case there is nothing to stop us from bringing the two lines together).

Isotopic lines have the same location relative to the other lines in the interlacing. Hence, if a and b are isotopic lines and c and d are two other lines of the same interlacing, then the triples a, c, d and b, c, d have the same linking coefficient. Using this necessary condition for lines to be isotopic, we can easily show that in the interlacing on the right in Figure 24 the line l is not isotopic to m. In fact, the triple l, n, k has linking coefficient $+1$, while the triple m, n, k has linking coefficient -1.

It is clear that, given any two isotopic lines in an interlacing, an isotopy can be found which interchanges them and causes all of the other lines to end up in the same place as before. Hence, isotopy of lines in an interfacing is an equivalence relation, and the set of all lines in an interfacing is partitioned into isotopy equivalence classes. The left and center interlacings in Figure 24 each has only one equivalence class, while the right interlacing has two: the lines k and l are in one class, and m and n are in another.

If we choose one line from each equivalence class in an interlacing, then the isotopy type of the resulting interlacing does not depend on our choice of our choice of line in each equivalence class. This interlacing is called the *derived interlacing.*

It is useful to pass to the derived interlacing if it contains fewer lines than the original interlacing. In order to recover the original interlacing from the derived one, one needs a relatively small amount of additional information, namely, how many lines were in each class and how they were linked to one another. In fact, by means of an isotopy one can reduce the original interlacing to a state in which the lines of each equivalence class are generatrices of the same family on a one-sheeted hyperboloid, and the hyperboloids containing the lines of the different equivalence classes do not intersect. We leave it as an exercise to construct an isotopy that does this.

The derived interlacing determines the relative location of the hyperboloids. To recover the original interlacing it remains only to specify one of the two families of generatrices on each hyperboloid. Here one does not have to do this at all if the class has only one line or if there is only one class in all and it has two lines. Otherwise the choice of a family of generatrices can be specified by means of a numerical invariant $\varepsilon = \pm 1$ for each isotopy class of lines in the interlacing; this is defined to be the linking coefficient of the triple of lines a, b, x, whee a and b are lines in the equivalence class and x is any line distinct from a and b. We shall prove that *this invariant depends only on the class of lines isotopic to a and b*. The proof will use certain formulas in which we will use the following notation: the linking coefficient of lines a, b, c will be denoted by $lk(a, b, c)$.

LEMMA. *For any lines a, b, c, d one has*

$$lk(a, b, c)lk(a, b, d)lk(a, c, d)lk(b, c, d) = 1.$$

This identity follows immediately from the definition of the linking coefficient of a triple of lines as the product of the linking coefficients of the three pairs of lines in the triple furnished with certain orientations. If we give orientations to the lines a, b, c, d and then compute the left side of the above equality, we obtain the product of the squares of the linking coefficients of all possible pairs of lines a, b, c, d. ●

We now prove that $lk(a, b, x)$ does not depend on x when a and b are isotopic lines of the interlacing. Let y be any line of the interlacing which is distinct from a, b, x. By the lemma we have

$$lk(a, b, x) = lk(a, b, y)lk(a, x, y)lk(b, x, y).$$

Since the lines a and b are isotopic, we have $lk(a, x, y) = lk(b, x, y)$, and hence $lk(a, b, x) = lk(a, b, y)$.

It remains to show that $lk(a, b, x)$ does not depend on the choice of representatives a and b of the isotopy class of lines. In fact, if c is a line which is isotopic to a and distinct from b, then, as already proved, we have

$$lk(a, b, x) = lk(a, b, c) = lk(a, c, b) = lk(a, c, x). \quad ●$$

A class of isotropic lines of an interlacing whose invariant is ε $(= \pm 1)$ will be called an *ε-class*.

Some interlacings can be brought to the form of an interlacing of one line by successively taking the derived interlacing. Such an interlacing is said to be *completely decomposable*. A completely decomposable interlacing can be characterized up to isotopy by the invariants associated with each transition from an interlacing to its derived interlacing. We shall introduce some notation for this characterization. An interlacing of p generatrices of a hyperboloid which form an ε-class of isotopic lines will be denoted by $\langle \varepsilon p \rangle$.

We now consider p hyperboloids which encompass disjoint regions and which have the lines of the interlacing $\langle \varepsilon p \rangle$ as their axes. An interlacing made up of p subinterlacings $A_1, \dots, A_p$, each of which is in the region bounded by the corresponding hyperboloid, will be denoted by $\langle +A_1, \dots, A_p \rangle$ if $\varepsilon = +1$ and $\langle -A_1, \dots, A_p \rangle$ if $\varepsilon = -1$. In situations where the signs do not matter to us, we shall omit them from the notation. For example, the interlacings in Figure 24 are characterized by the symbols $\langle +4 \rangle$, $\langle -4 \rangle$, and $\langle \langle +2 \rangle, \langle -2 \rangle \rangle$. The interlacings in Figure 16 are given by the symbols $\langle +4 \rangle$, $\langle +5 \rangle$, $\langle +6 \rangle$. The mirror interlacing of an even number p of lines that was constructed above is given by $\langle\langle +p/2\rangle, \langle -p/2\rangle\rangle$. In particular, the interlacing in Figure 17 is $\langle\langle +2\rangle, \langle -2\rangle\rangle$.

Not every interlacing is completely decomposable. For example, the derived interlacing for the interlacing in Figure 19 coincides with the original interlacing, and it cannot be placed on a hyperboloid (otherwise it would not be a mirror interlacing). This is the simplest example of an interlacing which is not completely decomposable.

Five lines

It can be shown (although it is not so easy as in the case of four lines) that any interlacing of five lines is isotopic to one of the seven interlacings shown in Figure 26. Six of them are nonmirror and completely decomposable; they are given by the following symbols:

$$\langle +5\rangle, \ \langle -5\rangle, \ \langle\langle +3\rangle, \langle -2\rangle\rangle, \ \langle\langle -3\rangle, \langle +2\rangle\rangle,$$
$$\langle +\langle 1\rangle, \langle -2\rangle, \langle -2\rangle\rangle, \ \langle -\langle 1\rangle, \langle +2\rangle, \langle +2\rangle\rangle.$$

The seventh is the interlacing in Figure 19. One can prove that the seven interlacings are not isotopic to one another by computing in each case the sum of the linking coefficients of the ten triples contained in the interlacing. The results are indicated under the diagrams in Figure 26. This sum is clearly preserved under isotopy, and we see that the sums for the seven interlacings are all different.

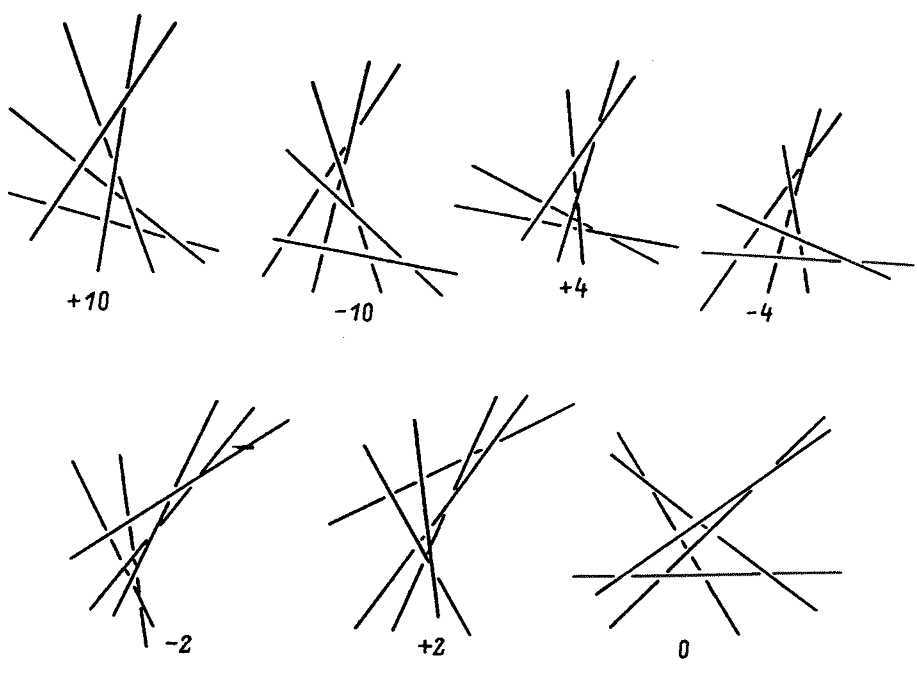


FIGURE 26

Six lines

It is by no means so easy to show that there are in all 19 types of interlacings of six lines (this theorem was proved by Mazurovskiĭ [2] in 1987). It is no longer possible to distinguish between nonisotopic interlacings using only the linking coefficients of the triples of lines in each interlacing. To prove that the isotopy classes are really distinct one has to perform computer calculations of more complicated invariants of the interlacings. Before describing Mazurovskiĭ's basic results in more detail, we give some definitions.

We shall need a construction proposed by Mazurovskiĭ to characterize interlacings of lines. Given a permutation σ, he constructs a corresponding interlacing defined up to isotopy. Let l and m be oriented skew lines whose linking coefficient is -1.

We mark off k points on each line l and m, and denote them by $A_1, \dots, A_k$ and $B_1, \dots, B_k$, where moving form point to point with increasing indices takes us in the direction of the line's orientation. Now, given a permutation σ of $\{1, \dots, k\}$, we construct an interlacing of k lines by joining A_i to $B_{\sigma(i)}$. Following Mazurovskiĭ, we shall denote this interlacing of the k lines $A_1B_{\sigma(1)}, \dots, A_kB_{\sigma(k)}$ by the symbol $hc(\sigma)$. Interlacings which are isotopic to an interlacing constructed in this way are said to be *isotopically horizontal*. This terminology comes from the fact that such an interlacing is isotopic to an interlacing which is made up of lines in parallel (and we may suppose horizontal) planes.

EXERCISE. Which of the interlacings encountered above are isotopically horizontal? Show that all interlacings of five or fewer lines are isotopically horizontal.

Mazurovskiĭ [8] showed that, if we want to prove that two interlacings of six lines are not isotopic or if we want to determine the isotopy class of a given interlacing of six lines, it is sufficient to use the polynomial invariant of equipped links in $\mathbf{R}P^3$ which was recently introduced by Drobotukhina. This invariant generalizes the Kauffman polynomial of links in $\mathbf{R}^3$. Here we shall not give a definition of this invariant, instead referring the reader to a forthcoming issue of this journal, in which Drobotukhina's article will appear.

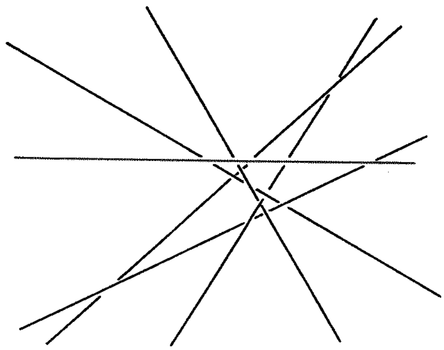
We return to the classification of interlacings of six lines. Of the 19 types, 15 consist of isotopically horizontal interlacings. The remaining four are the interlacing types M and L in Figures 27 and 28 (next page), and their mirror images M' and L'. Here M and its mirror image M' cannot be distinguished by means of the linking coefficients of the triples in the interlacings. But they can be distinguished using Drobotukhina's polynomial invariant, which for M is equal to

$$-A^{15} + 6A^{11} + 6A^9 - 5A^7 - 6A^5 + 10A^3 + 16A + A^{-1} - 10A^{-3} + 10A^{-7} + 5A^{-9},$$

and for M' is equal to

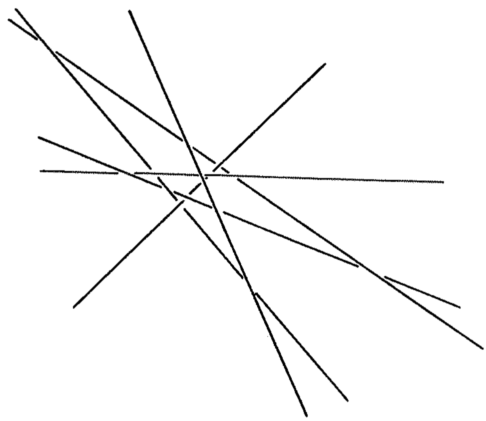
$$5A^9 + 10A^7 - 10A^3 + A + 16A^{-1} + 10A^{-3} - 6A^{-5} - 5A^{-7} + 6A^{-9} + 6A^{-11} - A^{-15}.$$

Similarly, L cannot be distinguished from the interlacing $hc(1, 2, 5, 6, 3, 4)$ by means of the linking coefficients, but these two interlacings do have different



Interlacing M

FIGURE 27



Interlacing L

FIGURE 28

polynomial invariants: for L it is

$$3A^{11}+8A^{9}+A^{7}-12A^{5}-A^{3}+22A+15A^{-1}-12A^{-3}$$
$$-13A^{-5}+10A^{-7}+15A^{-9}-5A^{-13}+A^{-17}$$

and for $hc(1, 2, 5, 6, 3, 4)$ it is

$$A^{13}+3A^{11}+2A^{9}+3A^{5}+5A^{3}+2A+3A^{-3}+7A^{-5}+4A^{-7}+A^{-11}+A^{-13}.$$

The derived interlacing of L coincides with L itself. The same holds for the mirror image L' of L, the interlacings M and M', and also the mirror interlacing $hc(1, 3, 5, 2, 6, 4)$. The interlacings $hc(1, 2, 4, 6, 3, 5)$ and $hc(5, 3, 6, 4, 2, 1)$ (which are mirror images of one another) both have the same derived interlacing, namely, a certain mirror interlacing of five lines (whose derived interlacing coincides with the original interlacing of five lines). The remaining types of interlacings of six lines are completely decomposable. Of those 12 types, two are the mirror interlacings

$$\langle\langle +3\rangle, \langle -3\rangle\rangle = hc(1, 2, 3, 6, 5, 4)$$

and

$$\langle\langle -\langle 1\rangle, \langle +2\rangle\rangle, \langle +\langle 1\rangle, \langle -2\rangle\rangle\rangle = hc(1, 2, 4, 6, 5, 3),$$

and the ten others can be divided into pairs of nonmirror interlacings, each pair consisting of an interlacing and its mirror image:

$$\langle +6\rangle = hc(1, 2, 3, 4, 5, 6), \qquad \langle -6\rangle = hc(6, 5, 4, 3, 2, 1);$$

$$\langle\langle +4\rangle, \langle -2\rangle\rangle = hc(1, 2, 3, 4, 6, 5),$$
$$\langle\langle +2\rangle, \langle -4\rangle\rangle = hc(5, 6, 4, 3, 2, 1);$$

$$\langle -\langle +3\rangle, \langle +2\rangle, \langle 1\rangle\rangle = hc(1, 2, 3, 5, 6, 4),$$
$$\langle +\langle -3\rangle, \langle -2\rangle, \langle 1\rangle\rangle = hc(4, 6, 5, 3, 2, 1);$$

$$\langle +\langle +2\rangle, \langle -2\rangle, \langle -2\rangle\rangle = hc(1, 2, 4, 3, 6, 5),$$
$$\langle -\langle +2\rangle, \langle +2\rangle, \langle -2\rangle\rangle = hc(5, 6, 3, 4, 2, 1);$$

$$\langle -\langle +2\rangle , \langle +2\rangle , \langle +2\rangle\rangle = hc(1, 2, 5, 6, 3, 4),$$
$$\langle +\langle -2\rangle , \langle -2\rangle , \langle -2\rangle\rangle = hc(4, 3, 6, 5, 2, 1).$$

Seven lines

The only thing known about the number of types of interlacings of seven lines is that it is large (Mazurovskiĭ showed that there are already 48 types of isotopically horizontal interlacings), and it is even. The number of types is even because any interlacing of seven lines is nonmirror.

Not only lines can be interlaced

We return to the definition of an interlacing of lines. We used this term to denote a finite set of pairwise skew lines in three-dimensional space. That is, among all possible sets of lines, we look at sets in general position which form an everywhere dense open subset of the space of all sets of lines.

The same can be done with other types of configurations. For example, we can consider finite sets of points in three-dimensional space. We say that such a set is *nonsingular* if for $k \leq 4$ there is no set of k points lying in a $(k-2)$-dimensional subspace (i.e., a four-tuple does not lie in a plane, a triple does not lie on a line, and all points are distinct). By an isotopy of such a set we mean a motion in the course of which these conditions are not violated. We say that a nonsingular set of points has the *mirror* property if it is isotopic to its mirror image.

We shall not treat the problem of classifying nonsingular sets of points, but rather turn our attention to the mirror problem.

THEOREM. *A nonsingular set of q points in three-dimensional space is nonmirror if $q \equiv 6 \bmod 8$ or $q \equiv 3 \bmod 4$ and $q \geq 7$.*

PROOF. Given a nonsingular set of points, we define s to be the sum of the linking coefficients of all triples of pairwise skew lines determined by pairs of points in our set. If our set has q points, then the number of such triples is $\binom{q}{2}\binom{q-2}{2}\binom{q-4}{2}/6$. If $q \equiv 6$ or $7 \bmod 8$, then this number is odd, and so s is also odd, since it is a sum of an odd number of terms each of which is ± 1. Clearly, s is preserved under isotopies of the set of points, and it is multiplied by -1 under mirror reflection. Hence, $s = 0$ for a mirror set. We conclude that if $q \equiv 6$ or $7 \bmod 8$, a nonsingular set of q points cannot have the mirror property. To treat the case $q \equiv 3 \bmod 8$, $q \geq 11$, we introduce another numerical invariant of a nonsingular set of points. We first note that, given any two points A and B of our configuration, one can determine two opposite cyclic orderings of the remaining $q-2$ points, namely, the order in which a plane rotating around the axis AB passes through them. If a triple of lines consists of the line AB along with two lines joining four successive points in this ordering (more precisely, one line joins the first point to the second and the other one joins the third point to the fourth), then we say that the triple is *cyclic.* Our numerical invariant of a nonsingular set of points will then be the sum of the linking coefficients of all cyclic triples of lines with distinguished first line. If $q \geq 7$, then there are $(q-2)\binom{q}{2}$ terms in this sum, and so the sum is odd if $q \equiv 3 \bmod 4$, $q \geq 7$. On the other hand, the sum is clearly equal to zero if we have a mirror set. ●

It is natural to ask questions about the mirror property for singular sets of points which are analogous to the four questions discussed above in connection with mirror interlacings of lines. We do not have complete answers to those questions.

In the same spirit one can consider a mixed situation: configurations of both lines and points. There ae various ways of defining a nonsingular configuration of this type, but the most natural definition is to require that the lines in the configuration be pairwise skew, the points not lie on the lines, and no two points lie in a common plane with one of the lines. Even less is known about the classification and mirror property of mixed configurations.

When investigating problems related to geometrical objects in Euclidean space, it is often useful to extend the space to a projective space. Projective space has even been called the "great simplifier". Passing to a projective space normally enables us to find a simpler projective classification problem inside our original classification problem, and this projective problem is usually interesting in its own right. The case of interlacings of lines is, however, an exception to this rule. When one goes from $\mathbf{R}^3$ to the projective space $\mathbf{R}P^3$, an interlacing of lines corresponds to a set of disjoint projective lines, and in this way one obtains all possible configurations of disjoint lines in $\mathbf{R}P^3$ in which no line is contained in the plane at infinity. Isotopy of interlacings is equivalent to the existence of an isotopy between the corresponding configurations of lines in $\mathbf{R}P^3$ in the course of which the lines remain disjoint.([1]) Here one need not concern oneself with the plane at infinity. Thus, the problem of classifying interlacings up to isotopy is actually equivalent to the corresponding problem for configurations of lines in projective space.([2]) We do not get a simpler problem. But in the case of the problem of classifying nonsingular sets of points in three-dimensional space, passing from $\mathbf{R}^3$ to $\mathbf{R}P^3$ leads to a splitting up of the problem; however, we shall not discuss this here.

Even in the case of interlacings of lines, passing to $\mathbf{R}P^3$ is not completely pointless. In $\mathbf{R}P^3$ we can see more clearly the topological reasons why interlacings are nonisotopic. As we showed at the very beginning of the article, any interlacing can be deformed into a set of parallel lines, and so there exists a homeomorphisms of $\mathbf{R}^3$ under which any interlacing is taken to any other given interlacing with the same number of lines. In $\mathbf{R}P^3$ this is no longer the case. The linking coefficient introduced above for oriented skew lines can be interpreted in terms of the usual linking coefficient in algebraic topology, applied to the corresponding lines in $\mathbf{R}P^3$ (except that we must double the topological invariant, which takes the values $\pm 1/2$, since for us the values ± 1 are more convenient). Moreover, in all cases we know of, the nonisotopy of two interlacings of lines is proved using topological invariants of the corresponding sets of projective lines in $\mathbf{R}P^3$, although there probably exist nonisotopic interlacings of lines for which the corresponding sets of projective lines can be taken to one another by means of a homeomorphism of the ambient space.

([1])This is explained by the fact that, in the space of all configurations of n disjoint lines in $\mathbf{R}P^3$, the configurations containing a line in the plane at infinity form a subset of codimension 2.

([2])Here are two other problems which are also equivalent: the problem of classifying sets of pairwise transversal two-dimensional subspaces in $\mathbf{R}^4$, with respect to motions under which they remain pairwise transversal two-dimensional subspaces; and the problem of classifying links in the sphere S^3 which are made up of great circles on the sphere, with respect to isotopies under which the circles remain disjoint great circles on S^3.

Perhaps we should show greater caution and make our definitions in accordance with the accepted topological terminology, i.e., call interlacings of lines isotopic if the corresponding sets of projective lines can be taken into one another by a homeomorphism of $\mathbf{R}P^3$ which is isotopic to the identity (recall that an isotopy of the homeomorphism $h: X \to Y$ is a family of homeomorphisms $h_t: X \to Y$ with $t \in [0, 1]$, $h_0 = h$, such that the map $X \times [0, 1] \to Y: (x, t) \mapsto h_t(x)$ is continuous). Then what we earlier called isotopies would be called *rigid isotopies.* Our cavalier attitude about this is permissible only because at the present level of knowledge we do not have examples of interlacings which show that these two types of isotopies actually lead to different equivalence relations. In some related situations, however, we do know such examples. We now discuss one such case.

Plane configurations of lines

At first glance it might seem that the world of configurations of lines in a plane resembles the world of configurations of lines in three-dimensional space, which we made an attempt to understand above. It is certainly easy to give definitions for plane configurations which are analogous to the basic definitions in this article. But, contrary to our expectations, these two worlds have very little in common.

Undoubtedly, the plane configuration analog of an interlacing of skew lines is a configuration of lines no three of which pass through a point and no two of which are parallel. The analog of an isotopy of interlacings is a motion during which the lines remain lines and the conditions on the location of the lines are preserved.

Passing from the plane to the projective plane changes the problem, and here, as usual, the projective problem turns out to be simpler and more elegant. In the projective problem the objects are sets of projective lines in $\mathbf{R}P^3$ which satisfy only one condition: no three of them pass through a point. Such a projective plane configuration of lines will be said to be *nonsingular.* A configuration of this type can also be interpreted as a set of planes through the origin in $\mathbf{R}^3$ such that no three of them contain a line.

In the case of nonsingular plane configurations of lines one must distinguish between isotopies and rigid isotopies. Two configurations are isotopic, or, equivalently, they have the same topological type, if one can be taken to the other by means of a homeomorphism $\mathbf{R}P^2 \to \mathbf{R}P^2$. Two configurations are said to be rigidly isotopic if they can be connected by a path in space whose points are nonsingular plane configurations of lines.

In the isotopic and rigid isotopic classification problems for plane configurations we do not have the mirror question. This is because the mirror image of any configuration is isotopic to the original configuration, since a reflection of the projective plane is isotopic to the identity map by means of an isotopy consisting of projective transformations. (More generally, the group of projective transformations of $\mathbf{R}P^2$ is connected.)

The isotopic and rigid isotopic classification problems for nonsingular plane configurations of lines have both been solved for configurations where the number of lines is ≤ 7, and in these cases the answer to both problems turns out to be the same (see Finashin [5]). If there are ≤ 5 lines, the isotopy type is determined by the number of lines. There are four types of nonsingular plane configurations of six lines, and 11 types of nonsingular plane configurations of seven lines. But when we reach configurations of more than seven lines, the isotopy and rigid isotopy classifications diverge sharply. Mnev [6] proved a surprising theorem, according to which, roughly speaking, a set of nonsingular plane configurations of lines which are isotopic to one another can have the homotopy type of any affine open semi-algebraic set, and, in particular, it can have any number of connected components, i.e., it can contain an arbitrary number of rigid isotopy classes. (This statement is imprecise, because in Mnev's work one considers ordered configurations in which the first four lines are in a fixed position; otherwise one must divide out by the action of the group of projective transformations.)

The simplest example known of nonsingular plane configurations of lines which are isotopic but not rigid isotopic can be found in Suvorov [7]. The configurations in this example have 14 lines.

Multidimensional generalizations of interlacings of lines

Thus, the theory of nonsingular plane configurations of lines is quite different from the theory of interlacings of lines. This closely corresponds to the picture one sees in the topology of manifolds: it is known that the topology of manifolds of successive dimensions has far fewer common features than the topology of manifolds whose dimensions differ by 4. In the topology of multidimensional manifolds one even has precise constructions which embed various parts of n-dimensional topology in $(n+4)$-dimensional topology. In surgery theory this construction is multiplication by a complex projective plane; in knot theory it is the two-fold covering of Bredon; and in the theory of singularities it is the addition to our function of the sum of the squares of two new variables. It seems that something similar occurs in the theory of projective configurations. Interlacings of skew lines in three-dimensional space appear to be related to configurations of pairwise skew $(2k-1)$-dimensional subspaces in $(4k-1)$-dimensional space. One can define a linking coefficient for oriented skew $(2k-1)$-dimensional subspaces of $(4k-1)$-dimensional space. Hence, all of the results on nonmirror interlacings that were proved using linking coefficients carry over to this multidimensional setting. Moreover, there is a simple construction which to any such configuration associates a configuration of the same type with k increased by 1.

This construction preserves the linking coefficients, isotopic configurations are taken to isotopic configurations, and perhaps to some extent one has an embedding of the theory of configurations of $(2k-1)$-dimensional subspaces of $(4k-1)$-dimensional space in the theory of configurations of $(2k+1)$-dimensional subspaces of $(4k+3)$-dimensional space. This gives rise to the possible development of a stable theory of projective configurations.(3)

Here we shall only give a description of this construction. As far as we know, it has not been published before, and it is the only original result of this paper. Our construction of a suspension is applicable not only to configurations of

(3)REMARK IN PROOF. This possibly has started to become a reality: Mazurovskiĭ has shown that any configuration of $\leq 2k+4$ disjoint $(2k+1)$-dimensional subspaces of $\mathbf{R}P^{4k+3}$ is rigidly isotopic to the suspension of a configuration of $(2k-1)$-dimensional subspaces of $\mathbf{R}P^{4k-1}$, and, if there are $\leq 2k+2$ subspaces in the configurations, then rigid isotopy of the suspensions is equivalent to rigid isotopy of the original configurations of $(2k-1)$-dimensional subspaces of $\mathbf{R}P^{4k-1}$.

$(2k-1)$-dimensional subspaces of $(4k-1)$-dimensional space. It applies to any configuration of finitely many subspaces in projective space; it increases the subspace dimension by 2 and the ambient space dimension by 4.

We first recall the construction of the join of ordered configurations. Let $L_1, \dots, L_r$ be subspaces of $\mathbf{R}P^p$, and let $M_1, \dots, M_r$ be subspaces of $\mathbf{R}P^q$. We imbed $\mathbf{R}P^p$ and $\mathbf{R}P^q$ in $\mathbf{R}P^{p+q+1}$ as skew subspaces, and we let $K_1, \dots, K_r$ denote the subspaces of $\mathbf{R}P^{p+q+1}$ such that K_i is the union of all lines which intersect L_i and M_i. We call the configuration of subspaces $K_1, \dots, K_r$ the *join* of our two configurations.(4)

By the *suspension* of an arbitrary configuration $L_1, \dots, L_r$ of subspaces of $\mathbf{R}P^p$ we mean its join with a configuration of r generatrices of a (one-sheeted) hyperboloid in $\mathbf{R}P^3$ with positive linking coefficient (i.e., its join with the configuration of lines in $\mathbf{R}P^3$ corresponding to the interlacing which we denoted by $\langle +r\rangle$). Since any two lines of the interlacing $\langle +r\rangle$ are isotropic, it follows that one can find an isotopy of this interlacing which permutes the lines in an arbitrary way. Hence, the join with an ordered configuration of subspaces $L_1, \dots, L_r$ in $\mathbf{R}P^p$ does not depend on the order. Thus, the suspension is well defined (up to rigid isotopy) for unordered configurations.

Connection with real algebraic surfaces of degree 4. Almost everything in the first two-thirds of the article concerning interlacings of lines, as well as everything concerning nonsingular sets of points in three-dimensional space, was published by Viro in 1985 in the note [1]. Interest in this subject was stimulated by work of Kharlamov on the classification of nonsingular real projective algebraic surfaces of degree 4 up to rigid isotopy (by which one means isotopies consisting of nonsingular algebraic surfaces). Earlier, a coarser classification of such surfaces up to mirror reflections and rigid isotopies was found by Nikulin [4]; and Kharlamov, using a very complicated technique which involved passing to the complex domain, proved that certain surfaces are nonmirror, in the sense that they are not rigid isotopic to their mirror images. It would be worthwhile to find an elementary proof.

Some of these surfaces decompose in the ambient three-dimensional space into a one-sheeted hyperboloid with handles and a number of separate spheres (the sum of the number of handles and the number of spheres is at most ten, and there are other restrictions, but we shall not dwell on this). From Harnack's theorem on the number of components of a plane curve it follows that every plane intersects at most three of the spheres of this surface. Hence, if we choose one point on each sphere, we obtain a nonsingular set of points whose isotopy type is determined by the surface, and a rigid isotopy of the surface corresponds to an isotopy of the set of points. Thus, if there are six or seven spheres, the surface must be nonmirror. In a similar way Kharlamov proved that many other degree 4 surfaces are nonmirror and completed the classification of nonsingular surfaces of degree 4 (see [3]). However, he was able to prove that certain of the surfaces are nonmirror only by passing to the complex domain and using the full theory of K3-surfaces.

(4)We have already encountered this construction. The isotopically horizontal interlacings introduced above (when we treated interlacings of six lines) are essentially the joins of sets of points on a line.

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Received 15/FEB/89

Translated by N. KOBLITZ