EXTENSIONS OF THE GUDKOV-ROHLIN CONGRUENCE

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§ 1. INTRODUCTION

1.1. The subject of the paper.

What pictures on the real projective plane $\mathbb{R}P^2$, up to homeomorphism, can be realized by a real algebraic curve ? The answer is not dufficult, unless we put a restriction on the degree of the curve (or a restriction of some other kind on the complexity of its equation).

However, for a fixed degree the question is very difficult and far from being solved in the complete generality, see e.g. G.Wilson [24] and O.Viro [21] (as for the other restrictions, see A.G.Khovansky [10]). The most complicated situation appears if the number of branches is great enough. Curves which have the maximal number of branches for a given degree (so called M-curves) are most remarkable from the topological point of view. It is the Gudkov-Rohlin congruence that makes one of the main features of the topology of M-curves of even degree.

The notion of M-curve, the Gudkov-Rohlin congruence, as well as many other results on nonsingular plane curves, permit appropriate extensions to the case of real algebraic manifolds of higher dimensions and to the case of real algebraic varieties (i.e. manifolds with singular points). Generalization of the notion of M-curve and the Gudkov-Rohlin congruence to the case of nonsingular real algebraic manifolds of arbitrary dimension were given by V.A.Rohlin [17] , [18] . Some extensions of the Gudkov-Rohlin congruence to the singular case were outlined in our note [9] . The present paper is devoted to extension of the Gudkov-Rohlin congruence and some related theorems to the singular case. Our results are fairly complete for plane curves, but higher dimensions appear only incidentally.

1.2. The Gudkov-Rohlin congruence and related ones.

Let A be a nonsingular plane projective real algebraic curve of degree m . It is said to be of *type I* or *dividing* if its real point set $\mathbb{R}A$ bounds in its complex point set $\mathbb{C}A$ (in this case $\mathbb{R}A$ divides $\mathbb{C}A$ into two parts, which are interchanged by the complex conjugation $\mathrm{conj}: \mathbb{C}P^2 \longrightarrow \mathbb{C}P^2 : (z_0 : z_1 : z_2) \longmapsto (\bar{z}_0 : \bar{z}_1 : \bar{z}_2)$ Otherwise it is said to be of *type II* or *non-dividing*. Below in this section the degree m of A is even, $m = 2k$. Then $\mathbb{R}A$ divides $\mathbb{R}P^2$ into two parts having $\mathbb{R}A$ as their common boundary. Only one of the parts is orientable; we denote it by $\mathbb{R}P^2_+$. The non-orientable part is denoted by $\mathbb{R}P^2_-$

By the well-known Harnack inequality [24] the number of components of $\mathbb{R}A$ is not more than $\frac{(m-1)(m-2)}{2}+1$. If it equals $\frac{(m-1)(m-2)}{2}+1$ then A is called an *M-curve*; if it equals $\frac{(m-1)(m-2)}{2}+1-i$ then A is called an *$(M-i)$-curve*.

(1.A) *If A is an M-curve, then*

$$\chi(\mathbb{R}P^2_+) \equiv k^2 \mod 8 \tag{1}$$

That is the Gudkov-Rohlin congruence. It was conjectured by D.A. Gudkov. He proved it for $m=6$ in [5]. The weakened congruence $\chi(\mathbb{R}P^2_+) \equiv k^2 \mod 4$ under a weaker hypethesis (see 1.D below) was proved by V.I.Arnold [1]. To the full extent it was proved by V.A.Rohlin [17].

There are several related congruences (also for a nonsingular A). We formulate three of them as (1.B) - (1.D). For the others, see Viro's survey [21] and the original papers by V.V.Nikulin [13] and T.Fiedler [4].

(1.B) *If A is an $(M-1)$-curve, then*

$$\chi(\mathbb{R}P^2_+) \equiv k^2 \pm 1 \mod 8 \tag{2}$$

(1.C) *If A is an $(M-2)$-curve of type* II, *then*

$$\chi(\mathbb{R}P^2_+) \equiv k^2 \quad \text{or} \quad k^2 \pm 2 \mod 8 \tag{3}$$

(1.D) *If A is a curve of type* I, *then*

$$\chi(\mathbb{R}P^2_+) \equiv k^2 \mod 4 \tag{4}$$

Proofs of (1.A)-(1.D) are reproduced below in 6.1. First, (1.B) was proved by D.A.Gudkov and A.D.Krahnov [6] and V.M.Kharlamov [8] independently, (1.C) by V.M.Kharlamov, see [19, 3.4], and A.Marin [12] independently; (1.D) is due to V.I.Arnold [1].

1.3. Two approaches.

Three proofs of the Gudkov-Rohlin congruence have been published. They are due to V.A.Rohlin [16], [17] and A.Marin [12]. The first [16] contains a mistake. The third [12] appears to be an improvement of the first. The example considered by Marin [12] seems to show that there is no correct proof of (1.A) which is closer to Rohlin's arguments [16] than Marin's proof.

Marin's [12] and Rohlin's second [17] approaches based on quite different techniques. Rohlin's proof works in any dimension while no generalization of Marin's proof to higher dimensions is known. Nevertheless the approaches seem to be closely related. Rohlin asked his students to find a relation and said that an understanding of it might lead to essential progress.

Both approaches admit extension to the case of singular curves. We did not seek identification of the results in their complete generality obtained for singular curves by those two approaches, although for all concrete situations considered the results coincide. Marin's approach seems to be simpler for our purposes, so we adopt it as the basic one. Rohlin's approach also has some important advantages. First, it is applicable to real algebraic varieties of arbitrary dimension; second, for some classes of singularities it gives results, which are more easy to formulate and use. In the last part of the paper we discuss these topics.

1.4. Two levels of results.

Our extensions of the Gudkov-Rohlin congruence, as many other statements on the topology of singular curves, involve some characteristics of the curve singularities. For efficient formulation of these results some additional investigation of the singularities is to be done. Due to a great diversity of singularities it is impossible to do this work once for all cases. Thus we distinguish two levels of results: first, general theorems (see § 3), which involve curves of vast classes and rather complicated characteristics of singularities (introduced in 2.3), and second, efficient theorems on curves of more special classes with singularities of some special types, formulations in this case involve only simplest characteristics of singularities (see § 4). The results of the first level are useful not only as initial steps to the results of the second level. In applications it is sometimes sufficient to know that some congruence is to be satisfied, for its efficient statement is obvious from known examples. See A.B.Korchagin [11] and sections 4.1 - 4.4 below.

1.5. Acknowledgements.

G.M.Polotovsky's work [14] on splitting curves of degree 6 suggested that there must be some congruences for singular curves, which are close to the Gudkov-Rohlin congruence but can not be straightforwardly reduced to it. Our first results in this direction were met by D.A. Gudkov, G.M.Polotovsky, E.I.Shustin and A.B.Korchagin with a stimulating interest. We are indebted to them for their encouragement.

§ 2. PREREQUISITE FOR STATING OF RESULTS

2.1. Preliminary arithmetics: $\mathbb{Z}/4$ -quadratic spaces.

By $\mathbb{Z}/4$-quadratic space we mean a triple $(V, \circ, q)$ consisting of a finite-dimensional vector space V over $\mathbb{Z}/2$, a symmetric bilinear form $V \times V \to \mathbb{Z}/2 : (x,y) \mapsto x \circ y$ and a function $q : V \to \mathbb{Z}/4$, which is quadratic with respect to that bilinear form, i.e.

$$q(x+y) = q(x) + q(y) + 2\cdot x\circ y \tag{5}$$

for $x,y \in V$, where $2\cdot : \mathbb{Z}/_2 \to \mathbb{Z}/_4$ is the unique non-zero homomorphism. The bilinear form $\circ$ is certainly determined by q via (5).

A $\mathbb{Z}/_4$ -quadratic space $Q=(V,\circ,q)$ is said to be nonsingular if its bilinear form $\circ$ is nonsingular, i.e. its radical $R(Q)=\{x\in V \mid \forall y\in V \ \ x\circ y=0\}$ is the zero-subspace. We say that a $\mathbb{Z}/_4$ -quadratic space $Q(V,\circ,q)$ is informative, if q vanishes on $R(Q)$. In this case $\circ$ and q induce well-defined bilinear and quadratic forms on $V/R(Q)$. The $\mathbb{Z}/_4$ -quadratic space appeared is nonsingular and it is called a nonsingular $\mathbb{Z}/_4$ -quadratic space associated with Q.

The isomorphism classes of nonsingular $\mathbb{Z}/_4$ -quadratic spaces form a commutative semigroup under the orthogonal sum operation. To obtain a group, one introduces the relation $(V,\circ,q)$ for any $\mathbb{Z}/_4$ -quadratic space $(V,\circ,q)$ with V containing a vector subspace H such that $\dim H = \frac{1}{2}\dim V$ and $q|_H = 0$ (and consequently $H\circ H=0$). The resulting factor-group is called the Witt group $WQ(\mathbb{Z}/_2,\mathbb{Z}/_4)$. It is isomorphic to $\mathbb{Z}/_8$ (see e.g. [2]). The isomorphism is set up by the van der Blij-Brown invariant $(V,\circ,q)\longmapsto B(q)$ defined by the formula

$$\exp\left(\frac{i\pi B(q)}{4}\right) = 2^{-\frac{\dim V}{2}} \sum_{x\in V} \exp\left(\frac{i\pi q(x)}{2}\right) \tag{6}$$

see e.g. L.Guillou and A-Marin [7].

Nonsingular $\mathbb{Z}/_4$ -quadratic spaces which determine the same element of $WQ(\mathbb{Z}/_2,\mathbb{Z}/_4$ are said to be cobordant. Informative $\mathbb{Z}/_4$ -quadratic spaces with cobordant associated nonsingular $\mathbb{Z}/_4$ -

quadratic spaces are also said to be cobordant. If $Q = (V, \circ, q)$ is an informative $\mathbb{Z}/_4$ -quadratic space, then the van der Blij-Brown invariant of its associated nonsingular $\mathbb{Z}/_4$ -quadratic space is denoted by $B(q)$. It can be calculated by the formula

$$\exp\left(\frac{i\pi B(q)}{4}\right) = 2^{-\frac{\dim V + \dim R(Q)}{2}} \sum \exp\left(\frac{i\pi q(x)}{2}\right) \tag{7}$$

2.2. Preliminary topology: the Rohlin-Guillou-Marin form.

Let X be an oriented smooth compact four-dimensional manifold, let F be its smooth compact two-dimensional submanifold (not necessarily orientable) with $\partial F = F \cap \partial X$ such that $in_* H_1(F; \mathbb{Z}/_2) = \{0\} \subset H_1(X; \mathbb{Z}/_2)$ (as usual in = inclusion), and let F realize in $H_2(X, \partial X; \mathbb{Z}/_2)$ the class which is the Poincaré dual to the Stiefel-Whitney class $w_2(X) \in H^2(X; \mathbb{Z}/_2)$.

Then there is a natural function $q: H_1(F; \mathbb{Z}/_2) \to \mathbb{Z}/_4$,which is quadratic in the sense of 2.1 with respect to the intersection form $H_1(F; \mathbb{Z}/_2) \times H_1(F; \mathbb{Z}/_2) \longrightarrow \mathbb{Z}/_2$, see e.g. [7]. We call it the Rohlin-Guillou-Marin form of the pair (X, F) . This q may be defined as follows. To define $q(\alpha)$ for $\alpha \in H_1(F; \mathbb{Z}/_2)$, realize α by an embedded closed smooth curve $l \subset F$, span l by a surface $P \subset X$, which is normal to F at $l = \partial P$ and transversal at inner points. Consider on l a field of lines tangent to F and normal to l and denote by $\varkappa$ the obstruction to extending this field to a field of lines normal to P . Then

$$q(\alpha) = \varkappa + 2(\operatorname{Int} P \circ F) \bmod 4 \tag{8}$$

where by $\operatorname{Int} P \circ F$ we mean the $\bmod 2$ -intersection number.

We like to consider here a slinghtly more general situation allowing F to have a corner, which is a smooth curve transversal to ∂X. The definition of q is naturally generalized to this situation. One may obtain q by smoothing F and checking that the result is independent on the choice of the smoothing. However there is a clear direct generalization of the definition of q given above. For α represented by ℓ, which does not meet the corner, $q(\alpha)$ is defined exactly as above.

2.3. Singular point data.

Let $f: \mathbb{C}^2 \longrightarrow \mathbb{C}$ be a holomorphic function, which is real (in the sence that $f(\bar{x},\bar{y}) = \overline{f(x,y)}$ for $(x,y) \in \mathbb{C}^2$). Let p be its real isolated singular point with $f(p)=0$

In this section to any such situation we assign $\mathbb{Z}/_2$ -vector spaces L_p and $\tilde{L}_p$, a homomorphism $\nu_p : \tilde{L}_p \longrightarrow L_p$, a $\mathbb{Z}/_4$ -quadratic space $(V_p, \circ, q_p)$, a bilinear pairing $L_p \times V_p \longrightarrow \mathbb{Z}/_2 : (x,y) \longmapsto x \sqcap y$, a subspace W_p of V_p, a subspace X_p of W_p and homomorphism $\omega_p : L_p \longrightarrow V_p / W_p$ and $\chi_p : \tilde{L}_p \longrightarrow V_p$. These objects are involved in formulation of our main theorems. We shall call them singular point data of p.
We can reduce the number of them, but for this we'll be made to pay with more heavy calculation in applications. In the corresponding simplified versions of formulations (see (3.A) and (3.C)) only L_p, $(V_p, \circ, q_p)$, $\sqcap$ of the singular point data are involved.

Denote by Φ the curve defined by the equation $f(x,y)=0$ and let $\nu: \Phi^{\sim} \longrightarrow \Phi$ be a normalization. Set

$$L_p = H_1(\mathbb{R}\Phi, \mathbb{R}\Phi \setminus p \ ; \mathbb{Z}/_2),$$

$$\tilde{L}_p = H_1(\mathbb{R}\Phi^{\sim}, \mathbb{R}\Phi^{\sim} \setminus \nu^{-1}(p) ; \mathbb{Z}/_2),$$

$$\nu_p = \nu_* .$$

Let $\mathcal{D}$ be a ball in $\mathbb{C}^2$ centered at p and so small that the pair $(\mathcal{D}, \mathbb{C}\Phi \cap \mathcal{D})$ is homemorphic to the cone over $(\partial\mathcal{D}, \mathbb{C}\Phi \cap \partial\mathcal{D})$. Let $\varepsilon > 0$ be such that for any $t \in (0, \varepsilon]$ the curve defined by the equation $f(x,y) = -t$ is nonsingular and transversal to $\partial\mathcal{D}$. Denote this curve by Φ_t. Set

$$R = \{(x,y) \in \mathcal{D} \cap \mathbb{R}^2 \mid f(x,y) \geqslant -\varepsilon\}$$

see fig. 1

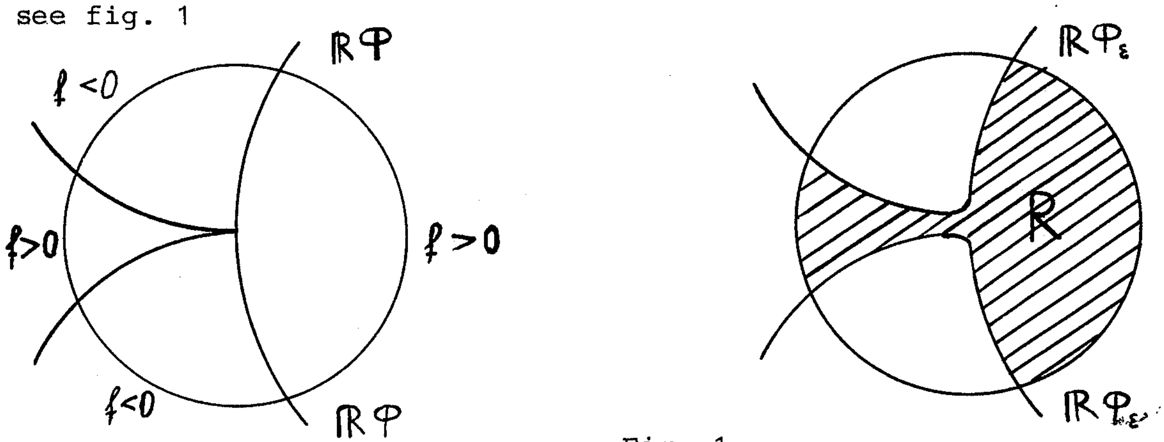

Fig. 1.

Now let us factorize by the complex conjugation $conj : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$. The ball $\mathcal{D}$ gives a ball $\mathcal{D}^* = \mathcal{D}/conj$. The surface R is not changed: it is contained in $fix(conj) = \mathbb{R}^2$ and so the natural projection $R \longrightarrow R/conj$ is a homeomorphism. We shall use the notation R for both R and $R/conj$. The surface $\mathbb{C}\Phi_\varepsilon \cap \mathcal{D}$ gives a compact surface $\mathcal{Y} = (\mathbb{C}\Phi_\varepsilon \cap \mathcal{D})/conj$ with a boundary $[(\mathbb{R}\Phi_\varepsilon \cap \mathcal{D}) \cup (\mathbb{C}\Phi_\varepsilon \cap \partial\Phi)]/conj$. The surfaces R and $\mathcal{Y}$ intersect in a curve $C = (\mathbb{R}\Phi_\varepsilon \cap \mathcal{D})/conj$, which is the common part of their boundaries. The union $\Sigma = R \cup \mathcal{Y}$ is a compact surface with a corner C.

The promised $\mathbb{Z}/4$-quadratic space $(V_p, \circ, q_p)$ is formed of $V_p = H_1(\Sigma; \mathbb{Z}/2)$, the intersection form $\circ$ of Σ and the Rohlin-Guillou-Marin form q_p of $(\mathcal{D}^*, \Sigma)$. As to the subspaces W_p and X_p they are nothing but in $in_* H_1(\mathcal{Y};\mathbb{Z}/_2)$ and in $in_* H_1(\partial\mathcal{Y}\setminus R;\mathbb{Z}/_2)$.

The promised pairing $\sqcap : L_p \times V_p \longrightarrow \mathbb{Z}/_2$ is defined by the intersection pairing

$$H_1(\Sigma,\partial\Sigma;\mathbb{Z}/_2)\times H_1(\Sigma;\mathbb{Z}/_2)$$

combined with a natural homomorphism

$$L_p = H_1(\mathbb{R}\Phi,\mathbb{R}\Phi\setminus p;\mathbb{Z}/_2)\xrightarrow[\text{(excision)}]{in_*^{-1}} H_1(\mathbb{R}\Phi\cap\mathcal{D},\mathbb{R}\Phi\cap\mathcal{D}\setminus p;\mathbb{Z}/_2)\xrightarrow{in_*^{-1}} H_1(\mathbb{R}\Phi\cap\mathcal{D},\mathbb{R}\Phi\cap\partial\mathcal{D};\mathbb{Z}/_2)\xrightarrow{in_*}$$
$$\longrightarrow H_1(\Sigma,\partial\Sigma;\mathbb{Z}/_2).$$

Since $\mathcal{Y}$ and Σ are connected, the factor-space $V_p/W_p =$ $= H_1(\Sigma;\mathbb{Z}/_2)/in_* H_1(\mathcal{Y};\mathbb{Z}/_2)$ is isomorphic to $H_1(\Sigma,\mathcal{Y};\mathbb{Z}/_2)$ and by excision, to $H_1(R,C;\mathbb{Z}/_2)$. To define ω_p, we combine these isomorphisms with the composition of the following isomorphisms

$$L_p = H_1(\mathbb{R}\Phi,\mathbb{R}\Phi\setminus p;\mathbb{Z}/_2)\xrightarrow[\text{(excision)}]{in_*^{-1}} H_1(\mathbb{R}\Phi\cap\mathcal{D},\mathbb{R}\Phi\cap\mathcal{D}\setminus p;\mathbb{Z}/_2)\xrightarrow{\partial}$$
$$\longrightarrow \tilde{H}_0(\mathbb{R}\Phi\cap\mathcal{D}\setminus p;\mathbb{Z}/_2)\xrightarrow{in_*^{-1}}\tilde{H}_0(\mathbb{R}\Phi\cap\partial\mathcal{D};\mathbb{Z}/_2)\xrightarrow{in_*}$$
$$\longrightarrow \tilde{H}_0(\{x\in\partial\mathcal{D}\cap\mathbb{R}^2 \mid 0\geq f(x)\geq-\varepsilon\};\mathbb{Z}/_2)\xrightarrow{in_*^{-1}}\tilde{H}_0(\partial C;\mathbb{Z}/_2),$$

homomorphism $in_*:\tilde{H}_0(\partial C;\mathbb{Z}/_2)\longrightarrow\tilde{H}_0(C,\mathbb{Z}/_2)$ and isomorphism $\partial^{-1}:\tilde{H}_0(C;\mathbb{Z}/_2)\longrightarrow H_1(R,C,\mathbb{Z}/_2)$. This definition is presented visually at fig. 2: given two components of $\mathbb{R}\Phi\cap\mathcal{D}\setminus p$ they determine an element of $H_1(\mathbb{R}\Phi\cap\mathcal{D},\mathbb{R}\Phi\cap\mathcal{D}\setminus p;\mathbb{Z}/_2)$, ω_p add to every component it adjycent arc of $\{x \in \partial \mathcal{D} \cap \mathbb{R}^2 \mid 0 \geq f(x) \geq -\varepsilon\}$ to give the element of $H_1(R, C; \mathbb{Z}/_2)$

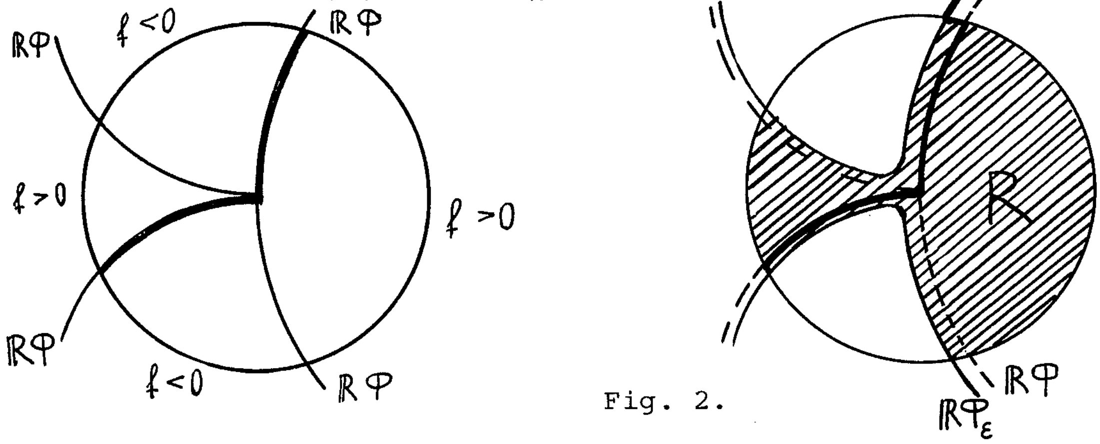

Fig. 2.

The group $H_1(\mathbb{R}\varphi^{\sim}, \mathbb{R}\varphi^{\sim} \setminus \nu^{-1}(p); \mathbb{Z}/_2)$ is generated by fundamental classes $[\ell]$ of components ℓ of $\mathbb{R}\varphi^{\sim} \cap \nu^{-1}(\mathbb{R}\varphi \cap \mathcal{D})$. For a component ℓ of $\mathbb{R}\varphi^{\sim} \cap \nu^{-1}(\mathbb{R}\varphi \cap \mathcal{D})$ both end points lie on one boundary circle of $\mathbb{C}\varphi^{\sim} \cap \nu^{-1}(\mathbb{R}\varphi \cap \mathcal{D})$. The image of the circle under ν is a boundary circle of $\mathbb{C}\varphi \cap \mathcal{D}$ and under the deformation $\mathbb{C}\varphi_t \cap \partial \mathcal{D}$, $0 \leq t \leq \varepsilon$ it remains to be the corresponding boundary circle C_t of the moving surface $\mathbb{C}\varphi_t \cap \mathcal{D}$. The image of C_t in $\partial \mathcal{D}^*$ is an arc C_t^* with end points lying in $\{x \in \partial \mathcal{D} \cap \mathbb{R}^2 \mid 0 \geq f(x) \geq -\varepsilon\}$. Hence, C_ε^* represents the element of $H_1(J, C; \mathbb{Z}/_2)$, which has the same boundary as $\omega_p(\nu_*[\ell]) \in$ $\in H_1(R, C; \mathbb{Z}/_2)$. By exactness of the sequence

$$0 \to H_1(\Sigma; \mathbb{Z}/_2) \to H_1(R, C; \mathbb{Z}/_2) \oplus H_1(J, C; \mathbb{Z}/_2) \to H_0(C; \mathbb{Z}/_2)$$

these elements determine an element of $H_1(\Sigma; \mathbb{Z}/_2)$. We set it to be $\chi_p([\ell])$.

2.4. Singular point diagram and its $\mathbb{Z}/_4$-quadratic spaces.

Let A be a reduced (i.e. without multiple components) plane projective real algebraic curve of degree $m = 2k$. Then its real point set $\mathbb{R}A$ divided $\mathbb{R}P^2$ into two parts having $\mathbb{R}A$ as their common boundary. Let us fix one of the parts and denote it by

$\mathbb{R}P^2_+$. The choice of the part is equivalent to choice, up to positive constant factor, of an equation $a = 0$ of the curve (here a is a real homogeneous polynomial of degree m). Since the sign is fixed, singular point data is well defined for each real singular point of the curve A.

The scheme of joining of real singular points by real branches is nothing but a one-dimensional graph. It will be denoted by Γ_A It can be thought of as $\mathbb{R}A$ with all the non-singular components deleted. We supply it by additional structures. The first one is the homomorphism $i : H_1(\Gamma_A ; \mathbb{Z}/_2) \longrightarrow H_1(\mathbb{R}P^2_+ ; \mathbb{Z}/_2)$ induced by the natural inclusions $\Gamma_A \hookrightarrow \mathbb{R}A \hookrightarrow \mathbb{R}P^2$, the second one-singular point data for each vertex of Γ_A and the third one-homomorphisms λ_p : $H_1(\Gamma_A ; \mathbb{Z}/_2) \longrightarrow L_p$ induced by the composition of the inclusion $\Gamma_A \hookrightarrow \mathbb{R}A$ and localization. The graph Γ_A supplied with these structures will be called the singular point diagram of the curve and will be denoted by Δ.

At the rest part of the section we assign to Δ two $\mathbb{Z}/_4$-qudratic spaces $\tilde{Q}_\Delta = (\tilde{V}_\Delta , \circ , \tilde{q}_\Delta)$ and $Q_\Delta = (V_\Delta , \circ , q_\Delta)$ and a subspace B_Δ of V_Δ . $\tilde{Q}_\Delta$ is involved in the simplified versions of the main formulations and does not involve $\tilde{L}_p , \varkappa_p , \nu_p , \chi_p$ and ω_p. It is well defined by the following

(i) $\tilde{V}_\Delta = H_1(\Gamma_A ; \mathbb{Z}/_2) \oplus \bigoplus_p V_p$

(ii) the restriction of $\circ$ to the summand $\bigoplus_p V_p$ is equal to the orthogonal sum of bilinear forms from singular point data, the restriction of $\circ$ to $H_1(\Gamma_A ; \mathbb{Z}/_2)$ is induced from the intersection form of $\mathbb{R}P^2$ via i :

$$x \circ y = i(x) \circ i(y)$$

and for $x \in H_1(\Gamma_A ; \mathbb{Z}/_2)$, $y \in V_p$

$$x \circ y = \lambda_p(x) \cap y \tag{9}$$

(iii) the restriction of $\tilde{q}_\Delta$ to $\bigoplus_p V_p$ is equal to the orthogonal sum $\bigoplus_p q_p$ of quadratic forms from the singular point data and the restriction of $\tilde{q}_\Delta$ to $H_1(\Gamma_A; \mathbb{Z}/_2)$ is expressed via i :

$$\tilde{q}_\Delta(x) = \begin{cases} (-1)^k & , \text{ if } i(x) \neq 0 \\ 0 & , \text{ if } i(x) = 0 \end{cases} \tag{10}$$

The $\mathbb{Z}/_4$ -quadratic space Q_Δ is a shortened substitute for $\tilde{Q}_\Delta$. It and B_Δ are not involved in the simplified versions of the main formulations and involve $L^{\sim}_p , \nu_p , \chi_p$ and ω_p . When simplified reading, one may omit them.

To define Q_Δ let us take the subspace of $\tilde{Q}_\Delta$ with the underlying space $V'_\Delta \subset V_\Delta$,

$$V'_\Delta = \{ x + \sum_p v_p \in H_1(\Gamma_A ; \mathbb{Z}/_2) \oplus \bigoplus_p V_p \mid \omega_p \lambda_p(x) = \\ = v_p \bmod W_p \quad \text{for each } p \},$$

and factor it by the following part of its radical :

$$R_\Delta = \{ x + \sum_p v_p \in V'_\Delta \mid v_p = 0 \quad \text{for each } p \}.$$

Thus $V_\Delta = V'_\Delta / R_\Delta$

To define B_Δ let us take

$$B'_\Delta = \left\{ \begin{array}{c} x + \sum_p v_p \in H_1(\Gamma_A; \mathbb{Z}/_2) \oplus \\ \oplus \bigoplus_p V_p \end{array} \middle| \begin{array}{l} \text{for each } p \text{ there exists} \\ x_p \in L^{\sim}_p \text{ such that } \nu_p(x_p) = \\ = \lambda_p(x), v_p - \chi_p(x_p) \in X_p \end{array} \right\}$$

and set $B_\Delta = B'_\Delta / B'_\Delta \cap R_\Delta$.

2.5. Extension of notions: M-curve, $(M-i)$-curve, types I and II.

Here we extend these notions (see 1.2) from nonsingular plane curves to general (not necessarily nonsingular and plane) curves.

A nonsingular real algebraic curve A is called an M-curve if the number of components of $\mathbb{R}A$ is equal to the genus of A enlarged by 1. For the given genus the number of components can not be more than in that case. The curve A is called an $(M-i)$-curve if the deficiency is equal to i. An irreducible singular curve is called an M-curve (respectively $(M-i)$-curve) if its nonsingular model (the result of normalization) is an M-curve (respectively $(M-i)$-curve). A reduced curve is called an M-curve if nonsingular models of all irreducible components are M-curves and is called an $(M-i)$-curve if the sum (over all irreducible components) of the deficiencies is equal to i.

A reduced real algebraic curve is said to be of type I if nonsingular models of all irreducible components are of type I. Otherwise it is said to be of type II.

§ 3. STATEMENT OF GENERAL RESULTS

3.1. Projective curves.

Let A be a reduced real plane projective curve of degree $m = 2k$ without non-real singular points and let $\mathbb{R}P^2_+$ be one of two parts of $\mathbb{R}P^2$ bounded by $\mathbb{R}A$. Let Δ be a singular point diagram of A related with $\mathbb{R}P^2_+$.

(3.A). Suppose the $\mathbb{Z}/4$-quadratic space $\tilde{\mathfrak{a}}_\Delta$ is informative. Let $\tilde{k}$ be zero, if $\mathrm{Int}\,\mathbb{R}P^2_+$ is orientable, and $\tilde{k} = (-1)^k$

otherwise. If A is an M-curve, then

$$\chi(\mathbb{R}P^2_+) \equiv k^2 + B(\tilde{q}_\Delta) + \tilde{b} \mod 8 \tag{11}$$

If A is an $(M-1)$-curve, then

$$\chi(\mathbb{R}P^2_+) \equiv k^2 \pm 1 + B(\tilde{q}_\Delta) + \tilde{b} \mod 8 \tag{12}$$

If A is an $(M-2)$-curve of type II, then

$$\chi(\mathbb{R}P^2_+) \equiv k^2 + d + B(\tilde{q}_\Delta) + \tilde{b} \mod 8 ,$$

where $d \in \{0, 2, -2\}$ (13)

If A is of type I, then

$$\chi(\mathbb{R}P^2_+) \equiv k^2 + B(\tilde{q}_\Delta) + \tilde{b} \mod 4 \tag{14}$$

We present another variant of this theorem. In all application it leads to the same results but usually through easier calculations.

(3.B). Suppose q_Δ vanishes on B_Δ. Let b be zero if $\mathbb{R}P^2_+$ is contractible in $\mathbb{R}P^2$ and $b = (-1)^k$ otherwise. If A is an M-curve then

$$\chi(\mathbb{R}P^2_+) \equiv k^2 + B(q_\Delta) + b \mod 8 \tag{15}$$

If A is an $(M-1)$-curve, then

$$\chi(\mathbb{R}P^2_+) \equiv k^2 \pm 1 + B(q_\Delta) + b \mod 8 \tag{16}$$

If A is an $(M-2)$-curve of type II, then

$$\chi(\mathbb{R}P^2_+) \equiv k^2 + d + B(q_\Delta) + b \mod 8, \tag{17}$$

where $d \in \{0, 2, -2\}$

If A is of type I, then

$$\chi(\mathbb{R}P^2_+) \equiv k^2 + B(q_\Delta) + b \mod 4 \tag{18}$$

3.2. Smoothings of a plane curve singularity.

As above in 2.3, let $f: \mathbb{C}^2 \longrightarrow \mathbb{C}$ be a real holomorphic function and p its real isolated singular point with $f(p)=0$. Denote by Φ the curve defined by the equation $f(x,y)=0$. Let $\mathcal{D}$ be a ball in $\mathbb{C}^2$ centered at p and so small that the pair $(\mathcal{D}, \mathbb{C}\Phi \cap \mathcal{D})$ is homeomorphic to the cone over $(\partial\mathcal{D}, \mathbb{C}\Phi \cap \partial\mathcal{D})$. Let $\varphi_t: \mathbb{C}^2 \longrightarrow \mathbb{C}$, $t \in \mathbb{R}$ be a continuous family of real holomorphic functions with $\varphi_0 = f$. Denote by Ψ_t the curve defined by the equation $\varphi_t(x,y)=0$. Suppose that $\mathbb{C}\Psi_t$ has no singular points in $\mathcal{D}$ and is transversal to $\partial\mathcal{D}$ for $t \in (0,\varepsilon]$. Set

$$\mathcal{D}_+ = \{(x,y) \in \mathcal{D} \cap \mathbb{R}^2 \mid \varphi_\varepsilon(x,y) \geqslant 0\}$$

In this section we state results on topology of $\mathcal{D}_+$ similar to (3.A- and (3.B). The main idea of the transfering is to glue pairs $(\mathcal{D}, \mathbb{C}\Psi_\varepsilon \cap \mathcal{D})$ and $(\mathcal{D}, \mathbb{C}\Phi \cap \mathcal{D})$ by an equivariant diffeomorphism of their boundaries arisen from the deformation $\mathbb{C}\Psi_t \cap \partial\mathcal{D}$, $t \in [0,\varepsilon]$. The gluing gives a 4-dimensional sphere with an involution and a subset which is a smooth submanifold at each point except one and is invariant under the involution. This situation is similar to that of the projective plane and a singular real curve in it. Moreover, we observe two simplifications: first, S^4 is simpler than $\mathbb{C}P^2$, second, here we have only one singular point.

Before stating the results we ought to describe modification of auxiliary notions (such as $\mathbb{Z}/4$ -quadratic space of the singular point diagram) involved in (3.A) and (3.B).

Let Γ be a bouquet of circles which are in 1 - 1 correspondence with components of $\mathbb{R}\Psi_\varepsilon \cap \mathcal{D}$ homeomorphic to I. It can be throught of as the union of these components of $\mathbb{R}\Psi_\varepsilon \cap \mathcal{D}$ glued to $\mathbb{R}\Phi \cap \mathcal{D}$ by the natural bijection of the boundaries. The number of the circles is denoted by τ , it is equal to the number of real branches of Φ passing through p .

Let $\lambda : H_1(\Gamma ; \mathbb{Z}/_2) \longrightarrow L_p$ be the composition

$$H_1(\Gamma ; \mathbb{Z}/_2) \xrightarrow{in_*} H_1(\Gamma , \Gamma \setminus p ; \mathbb{Z}/_2) \xrightarrow{in_*^{-1}} H_1(\mathbb{R}\Phi \cap \mathcal{D}, \mathbb{R}\Phi \cap \mathcal{D} \setminus p ;$$

$$\mathbb{Z}/_2) \xrightarrow{in_*} H_1(\mathbb{R}\Phi , \mathbb{R}\Phi \setminus p ; \mathbb{Z}/_2) = L_p$$

The graph Γ supplied with the singular point data of p and the homomorphism λ will be denoted by Δ . Now we assign to it $\mathbb{Z}/_4$ -quadratic spaces $\tilde{Q}_\Delta = (\tilde{V}_\Delta , \circ , \tilde{q}_\Delta)$ and $Q_\Delta = (V_\Delta , \circ , q_\Delta)$ and a subspace B_Δ of V_Δ , cf. 2.4. The space $\tilde{Q}_\Delta$ is involved in the simplified version of formulation and does not require $\tilde{L}_p$, χ_p , ν_p , χ_p and ω_p for its definition. It is well defined by the following

(i) $\tilde{V}_\Delta = H_1(\Gamma ; \mathbb{Z}/_2) \oplus V_p$

(ii) the restriction of $\circ$ to the summand V_p is the bilinear form from the singular point data.

$$x \circ y \qquad \text{for} \qquad x , y \in H_1(\Gamma ; \mathbb{Z}/_2),$$

$$x \circ y = \lambda_p(x) \cap y \qquad \text{for} \qquad x \in H_1(\Gamma ; \mathbb{Z}/_2) , y \in V_p \qquad (19)$$

(iii) the restriction of $\tilde{q}_\Delta$ to V_p is the quadratic form q_p

from the singular point data, the restriction of $\tilde{q}_\Delta$ to $H_1(\Gamma;\mathbb{Z}/_2)$ is equal to zero.

The $\mathbb{Z}/_4$ -quadratic space Q_Δ is a shortened substitute for $\tilde{Q}_\Delta$. Together with B_Δ it is not involved in the simplified statement. When simplified reading one may omit them.

To define Q_Δ let us take the subspace of $\tilde{Q}_\Delta$ with the underlying space $V'_\Delta \subset V_\Delta$

$$V'_\Delta = \{ x + v \in H_1(\Gamma;\mathbb{Z}/_2) \oplus V_p \mid \omega_p \lambda(x) = v \bmod W_p \}$$

and factor it by the following part of its radical :

$$R_\Delta = \{ x + v \in V'_\Delta \mid \omega_p \lambda(x) = 0 \}$$

Thus $V_\Delta = \tilde{V}_\Delta / V'_\Delta$. To define B_Δ let us take

$$B'_\Delta = \{ x + v \in H_1(\Gamma;\mathbb{Z}/_2) \oplus V_p \mid \exists\, y \in \tilde{L}_p : \nu_p(y) = \lambda(x)\ ,\ v - \chi_p(y) \in X_p \}$$

and set $B_\Delta = B'_\Delta / B'_\Delta \cap R_\Delta$

Now transfer the notions of $M;(M-i)$ -curve and type to the case of smoothings. A smoothing Ψ_ε of a singular point of Φ is called an M -smoothing, if the number of components of $(\mathbb{R}\Psi_\varepsilon \cap \mathcal{D}) \cup (\mathbb{C}\Psi_\varepsilon \cap \partial\mathcal{D})$ is equal to the genus (number of handles) of $\mathbb{C}\Psi_\varepsilon$ enlarged by 1. This number can not be more than in that case. The smoothing is called an $(M-i)$ -smoothing if the deficiency is equal to i . The smoothing is said to be of type I if $\mathbb{C}\Psi_\varepsilon \cap \mathcal{D}$ is divided by $\mathbb{R}\Psi_\varepsilon \cap \mathcal{D}$ into two path components. Otherwise it is said to be of type II .

(3.C). Suppose the $\mathbb{Z}/_4$ -quadratic space $\tilde{Q}_\Delta$ is informative. If Ψ is an M -smoothing, then

$$\chi(\mathcal{D}_+) \equiv B(\tilde{q}_\Delta) \bmod 8 \tag{20}$$

If Ψ_ε is an $(M-1)$-smoothing, then

$$\chi(\mathcal{D}_+) \equiv B(\tilde{q}_\Delta) \pm 1 \bmod 8 \tag{21}$$

If Ψ_ε is an $(M-2)$-smoothing of type II, then

$$\chi(\mathcal{D}_+) \equiv B(\tilde{q}_\Delta) + d \bmod 8 \quad \text{where } d \in \{-1, 1, -3\} \tag{22}$$

If Ψ_ε is of type I, then

$$\chi(\mathcal{D}_+) \equiv B(\tilde{q}_\Delta) \bmod 4 \tag{23}$$

We present another variant of this theorem. For all applications it leads to the same result but usually through easier calculations. Remind that τ involved below is the number of real branches of φ passing through p.

(3.D) Suppose q_Δ vanishes of B_Δ. If Ψ_ε is an M-smoothing, then

$$\chi(\mathcal{D}_+) \equiv B(q_\Delta) + \tau - 1 \bmod 8 \tag{24}$$

If Ψ_ε is an $(M-1)$-smoothing, then

$$\chi(\mathcal{D}_+) \equiv B(q_\Delta) + \tau - 1 \pm 1 \bmod 8 \tag{25}$$

If Ψ_ε is an $(M-2)$-smoothing of type II, then

$$\chi(\mathcal{D}_+) \equiv B(q_\Delta) + \tau + d \bmod 8, \text{ where } d \in \{-1, 1, -3\} \tag{26}$$