Докл. Акад. Наук СССР
Том 284 (1985), № 5

Soviet Math. Dokl.
Vol. 32 (1985), No. 2

TOPOLOGICAL PROBLEMS CONCERNING LINES AND POINTS OF THREE-DIMENSIONAL SPACE

UDC 512.77+515.16

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1. This note is concerned with how several lines or several points in general position with respect to one another can be arranged in three-dimensional space.

A *configuration of skew lines in the space* $\mathbf{R}^3$ is defined to be an (unordered) collection of nonoriented pairwise skew lines in $\mathbf{R}^3$. An isotopy of such a collection under which the lines remain pairwise skew lines is called a *rigid isotopy*. It can be regarded as a path in the space of configurations of a fixed number of skew lines (i.e., in the corresponding subspace of the symmetrized product of Grassmann manifolds $G_{4,2}$). A configuration of skew lines is called a *mirror configuration* if it is rigidly isotopic to a mirror image of itself. The problem naturally arises of classifying configurations of m skew lines to within rigid isotopies and of distinguishing the classes of mirror configurations in the set of classes of rigidly isotopic configurations of m skew lines. In particular, the question arises of determining those numbers m for which there exist mirror configurations of m skew lines.

It is convenient to pass from problems concerning lines in $\mathbf{R}^3$ to the equivalent problems concerning lines in the projective space $\mathbf{R}P^3$. A *nonsingular configuration of lines in* $\mathbf{R}P^3$ is defined to be a collection of nonoriented disjoint lines in $\mathbf{R}P^3$. From the algebraic-geometry point of view, a nonsingular configuration of lines in $\mathbf{R}P^3$ is a nonsingular curve in $\mathbf{R}P^3$ that splits into lines. The definitions of rigid isotopies and mirror configurations carry over in the obvious way to the case of nonsingular configurations of lines in $\mathbf{R}P^3$, and this leads to definitions agreeing with the general terminology in the topology of real algebraic manifolds (cf. [1] and [2]). The inclusion $\mathbf{R}^3 \to \mathbf{R}P^3$ determines an imbedding of the space of configurations of m skew lines in $\mathbf{R}^3$ into the space of nonsingular configurations of m lines in $\mathbf{R}P^3$. Since the image of this imbedding is the complement of a subspace of codimension 2, rigid isotopy of configurations of skew lines in $\mathbf{R}^3$ is equivalent to rigid isotopy of the corresponding nonsingular configurations in $\mathbf{R}P^3$.

The passage from $\mathbf{R}^3$ to $\mathbf{R}P^3$ enables us to perceive the topological reasons for the failure of configurations to be rigidly isotopic: nonsingular configurations of m lines in $\mathbf{R}P^3$ can fail to be rigidly isotopic because they are not topologically isotopic (see §2), while all configurations of m skew lines in $\mathbf{R}^3$ are topologically isotopic. Indeed, there exists an isotopy of an arbitrary configuration of skew lines making all its lines parallel: such an isotopy is supplied by an unbounded dilation of $\mathbf{R}^3$ with fixed plane transversal to all the lines of the original configuration.

A nonsingular configuration of lines in $\mathbf{R}P^3$ can be regarded as a collection of pairwise transversal two-dimensional vector subspaces of $\mathbf{R}^4$, and a rigid isotopy of it can be regarded as a motion of these subspaces under which they remain pairwise transversal two-dimensional subspaces. A nonsingular configuration of lines in $\mathbf{R}P^3$ can also be represented as a link formed by great circles in the sphere S^3.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 51A20; Secondary 51A05.

Along with nonsingular line configurations we can in the same spirit consider singular configurations, i.e., configurations of intersecting lines. They were intensively investigated in the heyday of projective geometry, but, as far as I know, rigid isotopies of them were not considered, nor were nonsingular configurations (the questions traditional in projective geometry are devoid of meaning for the latter). In the space of all configurations the nonsingular configurations constitute an open dense subset and are stable in the sense that each of them is rigidly isotopic to all sufficiently close configurations; hence the nonsingular configurations appear to merit being the primary objects of study.

A *nonsingular configuration of points* in $\mathbf{R}P^3$ is defined to be a finite collection of points in $\mathbf{R}P^3$, no k of which lie in a single projective subspace of dimension $k-2$, for $k \leq 4$.

It is not hard to generalize the above definitions to the definition of a nonsingular configuration of subspaces in $\mathbf{R}P^n$. I shall not consider this generalization here. Of the multidimensional configurations, the configurations of subspaces of dimension $2k-1$ and $k-1$ in $\mathbf{R}P^{4k-1}$ are the closest to those discussed below.

A *rigid isotopy* of a nonsingular configuration is defined to be an isotopy under which the configuration remains nonsingular. A nonsingular configuration is called a *mirror* configuration if it is rigidly isotopic to a mirror image of itself. These definitions lead to the same questions as discussed above in connection with line configurations.

2. Linking coefficients of lines. Two disjoint oriented lines L_1 and L_2 in $\mathbf{R}P^3$ have linking coefficient $\operatorname{lk}(L_1, L_2)$ equal to $+1$ or -1.(1) This coefficient can be defined, for example, as follows. Construct a plane P through L_1. The orientation of L_1 determines that of the complement $P \backslash L_1$. Its intersection index with L_2 is $\operatorname{lk}(L_1, L_2)$. Any integral chain C with $\partial C = 2L_1$ and transversal to L_2 can be used here instead of the plane P.

This definition presupposes a fixed orientation of the space $\mathbf{R}P^3$. If the orientation is reversed, or if the orientation of one of the lines is reversed, then $\operatorname{lk}(L_1, L_2)$ changes. The number $\operatorname{lk}(L_1, L_2)$ is preserved under isotopies of the union $L_1 \cup L_2$ in $\mathbf{R}P^3$.

Suppose now that $L = \{L_1, L_2, L_3\}$ is a nonsingular configuration of three lines in $\mathbf{R}P^3$, and let L_1^*, L_2^*, L_3^* be the same lines, endowed with some orientations. The product $\operatorname{lk}(L_2^*, L_3^*)\operatorname{lk}(L_1^*, L_3^*)\operatorname{lk}(L_1^*, L_2^*)$ is denoted by $\operatorname{lk}(L)$ and by $\operatorname{lk}(L_1, L_2, L_3)$. It clearly does not depend on the orientations of the lines L_i^*, is preserved under isotopies, and changes under reversal of the orientation of the space $\mathbf{R}P^3$.

3. Failure of a configuration to be a mirror configuration. For a nonsingular configuration X of lines in $\mathbf{R}P^3$ let $l(X)$ be the sum of the numbers $\operatorname{lk}(L)$, where L runs through the set of all subconfigurations of X consisting of three lines. Clearly $l(X)$ is preserved under isotopies of X and is multiplied by -1 when the orientation of $\mathbf{R}P^3$ is reversed; hence $l(X) = 0$ if X is a mirror configuration.

THEOREM 1. *Every nonsingular configuration of p lines in $\mathbf{R}P^3$ with $p \equiv 3 \bmod 4$ is not a mirror configuration.*

Indeed, $l(X) \equiv C_p^3 \bmod 2$; therefore, $l(X)$ is odd when $p \equiv 3 \bmod 4$.

THEOREM 2. *A nonsingular configuration of q points in $\mathbf{R}P^3$ is not a mirror configuration if $q \equiv 6 \bmod 8$ or if $q \equiv 3 \bmod 4$ and $q \geq 7$.*

The proof differs from that of the preceding theorem only in that the numerical invariants $s(Y)$ and $c(Y)$ of a nonsingular configuration Y of q points are used instead of $l(X)$. The invariant $s(Y)$ is the sum of the numbers $\operatorname{lk}(L)$ as L runs through the set of all nonsingular configurations of three lines determined by pairs of points in Y. There are 15 terms (equal to ± 1) in this sum; hence $s(Y)$ is odd if $q \equiv 6$ or $7 \bmod 8$. To define

(1)The usual definition gives $\pm 1/2$ (see, for instance, [3]). Here we consider the doubled linking coefficient.

$c(Y)$ we remark that any two points $A, B \in Y$ distinguish two opposite cyclic orders on the remaining $q-2$ points in Y: the orders in which they are encountered by the rotating plane containing A and B. A triple of lines the first of which joins A and B while the other two join four adjacent points in these cyclic orders (the first with the second, and the third with the fourth) is said to be cyclic. The number $c(Y)$ is the sum of the numbers $\mathrm{lk}(L)$, where L runs through the set of all triples with distinguished first lines. If $q \geq 7$, then there are $(q-2)C_q^2$ terms in this sum, and thus $c(Y)$ is odd if $q \equiv 3 \bmod 4$ and $q \geq 7$. On the other hand, if Y is a mirror configuration, then clearly $s(Y) = c(Y) = 0$.

4. The structure of a nonsingular line configuration. Let K be a nonsingular configuration of lines in $\mathbf{R}P^3$. Lines A, $B \in K$ are said to be adjacent if by means of a rigid isotopy of K we can put them on one side of some quadric and the remaining lines in K on the other side. Adjacent lines are obviously isotopic in the complement of the remaining lines. Lines A, $B \in K$ are said to be *homologous* if, endowed with some orientations, they realize a single integral homology class of the complement of the remaining lines in K. Lines A, $B \in K$ are homologous if and only if $\mathrm{lk}(A, C, D) = \mathrm{lk}(B, C, D)$ (or, what is equivalent, $\mathrm{lk}(A, B, C) = \mathrm{lk}(A, B, D)$) for any C, $D \in K$. A pair of homologous lines A, $B \in K$ is called an ε-pair, with $\varepsilon = \pm 1$, if $\mathrm{lk}(A, B, C) = \varepsilon$ for $C \in K \backslash \{A, B\}$. It is easily seen that adjacency and the property of being homologous are equivalences, that two lines are homologous if they are adjacent, and that if two homologous lines make up an ε-pair, then any two lines homologous to them also make up an ε-pair. A nonsingleton class of adjacent lines of a configuration is called an ε-class if any two lines in it constitute an ε-pair.

By a rigid isotopy it is possible to put the lines of each class of adjacent lines on a single quadric (as linear generators) and to make these quadrics bound disjoint regions in $\mathbf{R}P^3$. Each of these regions contracts to any line lying in it. A subconfiguration of K containing one line from each class of adjacent lines in K is called a *derived configuration* of K. It is easy to see that among adjacent line classes with representatives forming an ε-class in a derived configuration there is at most one that is a singleton class or an ε-class. A configuration is said to be *simple* if it coincides with a derived configuration of itself. A configuration is said to be *completely decomposable* if some multiple derived configuration of it consists of a single line. The inverse image of a completely decomposable configuration under the covering $S^3 \to \mathbf{R}P^3$ is a *cable link*. A completely decomposable configuration can be arranged by means of a rigid isotopy so that all its lines lie in regular neighborhoods of several linear generators of a quadric; lines lying in each of these neighborhoods are, in turn, in regular neighborhoods of several linear generators of a quadric lying in this neighborhood, and so on. We introduce the following notation for describing such a configuration (to within rigid isotopies). A configuration of p linear generators of a quadric that form ε-pairs is denoted by $\langle \varepsilon p \rangle$. The symbol $\langle +A_1, \ldots, A_r \rangle$ (respectively, $\langle -A_1, \ldots, A_r \rangle$) denotes a configuration of this kind whose lines are in regular neighborhoods of r linear generators of the quadric that form $(+1)$-pairs (respectively, (-1)-pairs) if the subconfigurations lying in these neighborhoods are denoted by $A_1, \ldots, A_r$.

5. Mirror configurations. For any k the configuration $\langle +\langle +k \rangle, \langle -k \rangle \rangle$ is clearly a mirror configuration. If k is even, then the addition to this configuration of a single line dividing each of its classes of adjacent lines into halves gives a mirror configuration of $2k+1$ lines. (We remark that in the case $k = 2$ this configuration is simple and that by proceeding in an analogous way with the configurations $\langle +\langle +2 \rangle, \langle -2 \rangle, \ldots, \langle -2 \rangle \rangle$ and $\langle +\langle 1 \rangle, \langle -2 \rangle, \ldots, \langle -2 \rangle \rangle$ we get simple configurations of p lines with any $p \geq 5$.) Thus, if $p \not\equiv 3 \bmod 4$, then there exists a mirror configuration of p lines (cf. §3).

6. Configurations of at most five lines. Every nonsingular configuration of p lines with $p \leq 4$ is completely decomposable and rigidly isotopic to one of the following configurations:

$$\langle 1 \rangle;\ \langle +2 \rangle;\ \langle +3 \rangle,\ \langle -3 \rangle;\ \langle +4 \rangle,\ \langle -4 \rangle, \langle +\langle +2 \rangle, \langle -2 \rangle \rangle.$$

Every nonsingular configuration of five lines is rigidly isotopic to either a simple mirror configuration described in §5 or one of the following completely decomposable configurations failing to be mirror configurations:

$$\langle +5 \rangle,\ \langle -5 \rangle;\ \langle +\langle +3 \rangle, \langle -2 \rangle \rangle,\ \langle +\langle +2 \rangle, \langle -3 \rangle \rangle;\ \langle +\langle 1 \rangle, \langle -2 \rangle, \langle -2 \rangle \rangle, \langle -\langle 1 \rangle, \langle +2 \rangle, \langle +2 \rangle \rangle.$$

No two of the configurations in this section are rigidly isotopic, because they differ in the number of lines or in the invariant l (see §3).

7. Insufficiency of the linking coefficients. Since nonsingular configurations of at most five lines are determined to within a rigid isotopy by the invariant l, the question arises as to whether it is possible to characterize a nonsingular line configuration to within rigid isotopies by the linking coefficients, i.e., whether nonsingular configurations K and K' are rigidly isotopic if there exists a bijection $\varphi \colon K \to K'$ such that $\mathrm{lk}(\varphi(A), \varphi(B), \varphi(C)) = \mathrm{lk}(A, B, C)$ for any $A, B, C \in K$. The answer is no. The configurations in the simplest counterexample known to me consist of ten lines and are obtained by adding two different pairs of lines to the configuration $\langle +\langle +\langle +2 \rangle, \langle -2 \rangle \rangle, \langle -\langle +2 \rangle, \langle -2 \rangle \rangle \rangle$. In both cases each of the lines added separates all the adjacent lines of the original configuration from each other, and the lines added are homologous to each other in the configurations obtained; hence, only the added lines are homologous to each other in the resulting configurations. However, in one case the lines added are adjacent, while in the other they are not only nonadjacent but even nonhomologous to each other in the complement of the remaining lines.

8. I directed my attention to the questions considered above under the influence of V. M. Kharlamov's account of their connection with the problem of classifying nonsingular surfaces of degree four to within rigid isotopies (see [2]). I take this opportunity to express to him my profound gratitude.

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Received 11/SEPT/84

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Translated by H. H. MCFADEN