CONSTRUCTION OF M-SURFACES

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1. Generalized Harnack Inequality and the Problem of Its Exactness. The well-known Harnack inequality [1], which says that the number of components of a nonsingular real algebraic curve of degree m in the projective plane does not exceed $(m^2 - 3m + 4)/2$, is generalized as follows: If A is a real algebraic variety and CA is its complexification, then $\dim H_*(A; Z_2) \leqslant \dim H_*(CA; Z_2)$ (see, e.g., [2]). If CA is a nonsingular hypersurface of degree m in the complex projective space CP^q, then the last inequality takes the form $\dim H_*(A; Z_2) \leqslant q + [(m-1)^q - (-1)^q](1 - m^{-1})$; in particular, for a surface in three-dimensional space,

$$\dim H_*(A; Z_2) \leqslant m^3 - 4m^2 + 6m. \quad (1)$$

Nonsingular real projective algebraic varieties for which $\dim H_*(A; Z_2) = \dim H_*(CA; Z_2)$ are called M-varieties.

Harnack [1] proved that this inequality is exact, i.e., that for every m there exists an M-curve in the projective plane of degree m. The sixteenth Hilbert problem specifically mentions the problem of the topology of M-curves. Investigations in this direction have led to the construction of a large number of M-curves and, on the other hand, to a proof of general theorems of Rokhlin [3] concerning the topology of M-varieties of arbitrary dimension. However, as far as the author knows very little information is available in the literature concerning the existence of M-varieties of dimensions $n \geqslant 2$; for example, the existence of M-surfaces of degree m in RP^3 is proved only for $m \leqslant 4$.

2. Main Result. In this note, we construct M-surfaces of arbitrary degree in RP^3.

THEOREM 1. For every natural number m, there exists in RP^3 a real nonsingular algebraic surface A_m of degree m with $(m^3 - 6m^2 + 11m)/6$ components, of which all except for one are homeomorphic to a sphere, the exceptional component for even m being homeomorphic to a sphere with $(2m^3 - 6m^2 + 7m)/6$ handles, while for odd m, it is homeomorphic to the projective plane with $(2m^3 - 6m^2 + 7m - 3)/6$ handles.

The construction of the surfaces A_m (see Sec. 4) generalizes the construction of M-curves of Harnack [1].

3. Exactness of the Strengthened Petrovskii–Oleinik Inequality. The preceding theorem proves not only the exactness of the generalized Harnack inequality, it also proves the exactness of the strengthened form proved by Kharlamov [4] of the left-hand inequality in the Petrovskii–Oleinik inequalities [5]

$$-(2m^3 - 6m^2 + 7m - 6)/3 \leqslant \chi(A) \leqslant (2m^3 - 6m^2 + 7m)/3. \quad (2)$$

This strengthening consists in the following: If A is a nonsingular real projective algebraic surface of degree $m \neq 2$ in RP^3, having k_+ components homeomorphic to a sphere and k_0 components homeomorphic to a torus, then

$$-(2m^3 - 6m^2 + 7m - 6)/3 \leqslant \chi(A) - 2(k_+ + k_0).$$

4. Construction of the Surfaces A_m. For the construction, we will need a series of plane curves $C_1, C_2, \ldots$, satisfying the following two conditions: (i) C_m is a curve of degree m intersecting some straight-line segment I_m in m points contained in a single component C_m^o of C_m; (ii) for odd m, all the ovals of the curve C_m lie outside one another, for even m the oval C_m^o includes $(m^2 - 6m + 8)/8$ ovals lying outside one another, and the remaining $(3m^2 - 6)/8$ ovals lie outside C_m^o and outside one another. Such a series of M-curves was constructed by Harnack [1].

For each m, we construct a convex quadrilateral having I_m as one of its sides and which intersects C_m along m arcs joining the side I_m to the opposite side. With the aid of projective transformations, we take all these quadrilaterals onto a single quadrilateral Q in such a way that the curves C_m with odd m intersect the base of Q while those C_m with even m intersect the lateral sides of Q.

Let $\gamma_m(x_0, x_1, x_2) = 0$ be the equation of the curve C_m and assume the polynomials γ_m are chosen so that on each side of Q, all the polynomials γ_m not taking the value 0 on the side take values of the same sign there.

We put $\alpha_1(x_0, x_1, x_2, x_3) = x_3 + t_1\gamma_1(x_0, x_1, x_2)$ for some $t_1 > 0$ and $A_1 = \{x \in RP^3 \mid \alpha_1(x) = 0\}$. Assume that the polynomial α_{m-1} and surface $A_{m-1} = \{x \in RP^3 \mid \alpha_{m-1}(x) = 0\}$ have already been constructed. We consider the family of surfaces $A_t^{(m)} = \{x \in RP^3 \mid x_3\alpha_{m-1} + t\gamma_m = 0\}$. For some $t_m > 0$, the surfaces $A_t^{(m)}$ with $t \in (0, t_m]$ are nonsingular and mutually isotopic. We put $\alpha_m = x_3\alpha_{m-1} + t_m\gamma_m$ and $A_m = \{x \in RP^3 \mid \alpha_m(x) = 0\}$.

5. Exactness of the Left-Hand Inequality of Petrovskii–Oleinik. The following theorem shows that the left-hand inequality in (2) is exact.

THEOREM 2. For every natural number m, there exists in RP^3 a real nonsingular algebraic surface A_m' of degree m homeomorphic for even m to a sphere with $(2m^3 - 6m^2 + 7m)/6$ handles and for odd m to a projective plane with $(2m^3 - 6m^2 + 7m - 3)/6$ handles.

The surfaces A_m' can be constructed in the same way as the A_m by taking in place of the C_m real rational curves satisfying condition (i) of Sec. 4 and such that all their singular points are isolated real simple double points. The existence of such curves is proved by means of a modification of Harnack's construction [1].

6. Other Results. The author has also used the same technique to: (i) construct other series of M-surfaces in RP^3 and series of M-surfaces in line bundles over RP^2; (ii) proved that in order for there to exist in RP^3 a nonsingular real algebraic surface A of degree m with $\dim H_*(A; Z_2) = b$, it is necessary and sufficient that $b \equiv m \pmod 2$ and $3(1 - (-1)^m)/2 \leqslant b \leqslant m^3 - 4m^2 + 6m$; (iii) constructed M-hypersurfaces in RP^4 of any degree. I conjecture that M-hypersurfaces of arbitrary degree can be constructed analogously in RP^q for any q.*

LITERATURE CITED

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*Remark Added in Proof. This conjecture has now been proved. Moreover, the author has succeeded in proving the same assertion for arbitrary complete intersections of projective space.

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