Stony Brook University
Mathematics Department
Oleg Viro

Geometry, Topology I
MAT 530
December 8, 2013

Program for the final exam

General topology

Topological structure in a set. Axioms of topological structure. Open and closed sets. Neighborhoods. Bases of a topological structure. Tests for a collection of subsets for being a base of a topology.

Continuous maps - definition and main properties. Topological structure induced by a map to or from a topological space.

Homeomorphisms - definition and main properties. Homeomorphic spaces. Homeomorphism problem and topological properties. Topological embeddings.

Subspaces of a topological space. Relativity of openness.

Metric spaces. Axioms of metric. Balls and spheres. Topological structure defined by a metric. Two descriptions of open sets in a metric space and their equivalence.

Local continuity and its relation to continuity. Local continuity in metric spaces and the ϵ-δ definition.

Quotient topology. Relations between topological properties of a topological space and its quotient space. Continuity of quotient maps.

Anatomy of a topological space: interior, boundary and exterior points; interior, exterior, closure and boundary of a set. Everywhere dense set. Test for everywhere dense set.

Fundamental covers. Continuity of a map continuous on elements of a fundamental cover.

Connectedness - definitions and their equivalence. Connected sets and their properties. Connected components. Connectedness and continuous maps. Connectedness of real line. Intermediate value theorem and its generalizations. Applications of connectedness to the homeomorphism problem.

Paths. Path connected spaces. Path connected sets. Path connected components. Path connectedness and continuous maps. Relation between connectedness and path connectedness.

Hausdorff axiom and uniqueness of limit. The first separation axiom and closedness of finite sets. Third and fourth separation axioms. Relations between separation axioms. Proofs of separation axioms for metric spaces.

Countability axioms. Second countability and separability. Lindelöf theorem. Second countability of a metric separable space. Bases at a point and first countability. Sequential approach to topology and continuity.

Compactness definitions in terms of open covers and collections of closed sets with finite intersection property. Compact sets and their properties. Relation between the properties of being closed and compact. Compactness and continuous maps. Compactness of a closed segment. Compactness in Euclidean space. Sequential compactness and its relation to compactness.

Examples of quotient spaces. Möbius strip, Klein bottle, basic surfaces. Descriptions of the real projective space $\mathbb{R}P^n$ and their equivalences.

Product of topological spaces. Topological properties of projections and fibers. Relations between topological properties of topological spaces and their product.

Fundamental group and coverings.

Homotopy as a map, as a family of maps and as a relation. Rectilinear homotopy. Stationary homotopy. Homotopy of paths. Multiplication of paths and their homotopy classes. Homotopy properties of path multiplication.

Definition of fundamental group and high homotopy groups. Fundamental group of a product. Simply-connectedness.

Dependence of fundamental group on the base point. Properties of translation maps.

Covering spaces. The number of sheets of a covering. Theorems on path lifting.

Calculations of fundamental group using universal coverings: for circle, projective spaces and bouquet of circles.

Behavior of fundamental group under a continuous map. Properties of homomorphisms of fundamental group induced by continuous maps. Retracts and fundamental groups. Borsuk theorem and Brouwer theorem. Homotopy equivalences and deformation retractions.

Injectivity of homomorphism induced by a covering projection. The group of a covering. Classification of coverings

Cellular spaces (cw-complexes). Homotopy classification of 1-dimensional cellular spaces. The Euler characteristic of a finite cellular space. Surjectivity of the homomorphism induced by inclusion of the 1-skeleton. The kernel of the homomorphism. Calculation of the fundamental group of a cellular space.

Manifolds.

Topological manifolds without and with boundary. Interior and boundary of a manifold as manifolds without boundary. General properties of manifolds (number of connected components, equivalence of connectedness and path connectedness). Manifold as the sum of its connected components. Product of manifolds. Dimension of manifold. The boundary of Euclidean half-space.

One-dimensional manifolds. Topological classification of 1-manifolds.

Triangulated two-dimensional manifolds. Topological classification of triangulated compact 2-manifolds. Orientations and orientability of a triangulated 2-manifold. Topological invariance of orientability for triangulated compact 2-manifolds. Topological and homotopy invariance of the Euler characteristic for closed 2-manifolds. Recognizing of the topological type of a compact 2-manifold.

Short list of topics

Some of the topics listed below will be included (in a rephrased form) into the exam. It will be required to formulate the relevant definitions and theorems, and provide a detailed proofs.

(1) Metric topology: various descriptions of open sets and their equivalence.

(2) Fundamental coverings and continuity of a map which has continuous restrictions to elements of a fundamental covering.
(3) Properties of connected components.
(4) Connectedness and convexity of a subset of real line. Which subsets of $\mathbb{R}$ are connected.
(5) Relations between second countability and separability.
(6) Compactness in metric spaces and in Euclidean spaces.
(7) Compactness and continuous maps.
(8) Special properties of continuous maps from a compact space to a Hausdorff space.
(9) The Lebesgue lemma.
(10) Relations between sequential compactness and compactness.
(11) Homeomorphism between a fiber of product and the factor.
(12) Relations between topological properties of topological spaces and their product.
(13) Relations between topological properties of a topological space and its quotient space.
(14) Descriptions of the real projective space $\mathbb{R}P^n$ and their equivalences.
(15) Homotopy properties of path multiplication.
(16) Dependence of fundamental group on base point. Properties of translation maps.
(17) The number of sheets of a covering.
(18) Theorems on path lifting.
(19) Calculation of the fundamental group of circle.
(20) Calculation of the fundamental group of a bouquet of circles.
(21) Properties of homomorphisms of fundamental group induced by a continuous maps.
(22) Retracts and fundamental groups. Borsuk theorem and Brouwer theorem.
(23) Homotopy equivalences and deformation retractions.
(24) Injectivity of homomorphism induced by a covering projection. The group of a covering.
(25) Homotopy classification of 1-dimensional cellular spaces.
(26) Properties of the inclusion homomorphism of 1-skeleton.
(27) Topological properties of manifolds.
(28) Topological properties of connected components of a manifold.
(29) Topological classification of 1-manifolds.
(30) Properties of triangulations of 2-manifolds.
(31) Operations with families of polygons representing compact 2-manifold. Reduction theorem for an irreducible family of polygons: statement and proof of homeomorphism between connected sum of three copies of the projective plane and connected sum of torus and projective plane.