Stony Brook University
Mathematics Department
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Geometry, Topology I
MAT 530
December 2, 2012

Program for the final exam

General topology

Topological structure in a set. Axioms of topological structure. Open and closed sets. Neighborhoods. Bases of a topological structure. Tests for a collection of subsets for being a base of a topology.

Metric spaces. Axioms of metric. Balls and spheres. Topological structure defined by a metric. Two descriptions of open sets in a metric space and their equivalence.

Subspaces of a topological space. Relativity of openness.

Interior, boundary and exterior points. Interior, exterior, closure and boundary of a set. Everywhere dense set. Test for everywhere dense set.

Continuous maps - definition and main properties. Local continuity and its relation to continuity. Fundamental covers. Continuity of a map continuous on elements of a fundamental cover.

Homeomorphisms - definition and main properties. Homeomorphic spaces. Homeomorphism problem and topological properties. Topological embeddings.

Connectedness - definitions and their equivalence. Connected sets and their properties. Connected components. Connectedness and continuous maps. Connectedness of real line. Intermediate value theorem and its generalizations. Applications of connectedness to the homeomorphism problem.

Paths. Path connected spaces. Path connected sets. Path connected components. Path connectedness and continuous maps. Relation between connectedness and path connectedness.

Hausdorff axiom and uniqueness of limit. The first separation axiom and closedness of finite sets. Third and fourth separation axioms. Relations between separation axioms. Proofs of separation axioms for metric spaces.

Countability axioms. Second countability and separability. Lindelöf theorem. Second countability of a metric separable space. Bases at a point and first countability. Sequential approach to topology and continuity.

Compactness definitions in terms of open covers and collections of closed sets with finite intersection property. Compact sets and their properties. Relation between the properties of being closed and compact. Compactness and continuous maps. Compactness of a closed segment. Compactness in Euclidean space. Sequential compactness and its relation to compactness.

Product of topological spaces. Topological properties of projections and fibers. Relations between topological properties of topological spaces and their product.

Quotient topology. Relations between topological properties of a topological space and its quotient space. Continuity of quotient maps. Examples of quotient spaces. Möbius strip, Klein bottle, basic surfaces. Descriptions of the real projective space $\mathbb{R}P^n$ and their equivalences.

Fundamental group and coverings.

Homotopy as a map, a family of maps and as a relation. Rectilinear homotopy. Stationary homotopy. Homotopy of paths. Multiplication of paths and their homotopy classes. Homotopy properties of path multiplication.

Definition of fundamental group and high homotopy groups. Fundamental group of a product. Simply-connectedness.

Dependence of fundamental group on the base point. Properties of translation maps.

Coverings. The number of sheets of a covering. Theorems on path lifting.

Calculations of fundamental group using universal coverings: for circle, projective spaces and bouquet of circles.

Behavior of fundamental group under a continuous map. Properties of homomorphisms of fundamental group induced by a continuous maps. Retracts and fundamental groups. Borsuk theorem and Brouwer theorem. Homotopy equivalences and deformation retractions.

Injectivity of homomorphism induced by a covering projection. The group of a covering. Classification of coverings

Cellular spaces (cw-complexes). Homotopy classification of 1-dimensional cellular spaces. The Euler characteristic of a finite cellular space. Surjectivity of the homomorphism induced by inclusion of the 1-skeleton. The kernel of the homomorphism. Calculation of fundamental group of a cellular space.

Manifolds.

Topological manifolds without and with boundary. Interior and boundary of a manifold as manifolds without boundary. General properties of manifolds (number of components, equivalence of connectedness and path connectedness). Manifold as the sum of its components. Product of manifolds. Dimension of manifold. The boundary of Euclidean half-space.

One-dimensional manifolds. Topological classification of one-dimensional manifolds.

Triangulated two-dimensional manifolds and their topological classification.

Short list of topics

Some of the topics listed below will be included (in a rephrased form) into the exam. It will be required to formulate the relevant definitions and theorems, and provide a detailed proofs.

(1) Metric topology: two descriptions of open sets and their equivalence.
(2) Relation between continuity and local continuity.
(3) Properties of connected components.
(4) Intermediate value theorem and its generalizations.
(5) Relation between connectedness and path-connectedness.
(6) Proofs of separation axioms for metric spaces.
(7) The Hausdorff axiom and its consequences.
(8) Second countability of a metric separable space.
(9) Sequential approach to topology and continuity.
(10) Relations between the properties of being closed and compact.
(11) Compactness and continuous maps.

(12) Compactness in Euclidean space.
(13) Homeomorphism between a fiber of product and the factor.
(14) Relations between topological properties of topological spaces and their product.
(15) Relations between topological properties of a topological space and its quotient space.
(16) Continuity of quotient maps.
(17) Descriptions of the real projective space $\mathbb{R}P^n$ and their equivalences.
(18) Homotopy properties of path multiplication.
(19) Dependence of fundamental group on base point. Properties of translation maps.
(20) The number of sheets of a covering.
(21) Theorems on path lifting.
(22) Calculation of the fundamental group of circle.
(23) Calculation of the fundamental group of a bouquet of circles.
(24) Properties of homomorphisms of fundamental group induced by a continuous maps.
(25) Retracts and fundamental groups. Borsuk theorem and Brouwer theorem.
(26) Homotopy equivalences and deformation retractions.
(27) Injectivity of homomorphism induced by a covering projection. The group of a covering.
(28) Homotopy classification of 1-dimensional cellular spaces.
(29) Properties of the inclusion homomorphism of 1-skeleton.
(30) Topological properties of manifolds.
(31) Topological classification of 1-manifolds.