SIMILARITY

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Euclidean Geometry can be described as a study of the properties of geometric figures, but not all conceivable properties. Only the properties which do not change under isometries deserve to be called geometric properties and studied in Euclidian Geometry.

Some geometric properties are invariant under transformations that belong to wider classes. One such class of transformations is similarity transformations. Roughly they can be described as transformations preserving shapes, but changing scales: magnifying or contracting.

The part of Euclidean Geometry that studies the geometric properties unchanged by *similarity transformations* is called the *similarity geometry.* Similarity geometry can be introduced in a number of different ways. The most straightforward of them is based on the notion of ratio of segments.

The similarity geometry is an integral part of Euclidean Geometry. However, its main notions emerge in traditional presentations of Euclidean Geometry (in particular in the textbook by Hadamard that we use) in a very indirect way. Below it is shown how this can be done according to the standards of modern mathematics. But first, in Sections 1 - 4, the traditional definitions for ratio of segments and the Euclidean distance are summarized.

1. Ratio of commensurable segments. If a segment CD can be obtained by summing up of n copies of a segment AB, then we say that $\dfrac{CD}{AB} = n$ and $\dfrac{AB}{CD} = \dfrac{1}{n}$.

If for segments AB and CD there exists a segment EF and natural numbers p and q such that $\dfrac{AB}{EF} = p$ and $\dfrac{CD}{EF} = q$, then AB and CD are said to be *commensurable,* $\dfrac{AB}{CD}$ is defined as $\dfrac{p}{q}$ and the segment EF is called a *common measure* of AB and CD.

The ratio $\dfrac{AB}{CD}$ *does not depend on the common measure* EF.

This can be deduced from the following two statements.

For any two commensurable segments there exists the greatest common measure.

The greatest common measure can be found by geometric version of the *Euclidean algorithm*, for an English translation of Euclid's text and its discussion from the modern viewpoint see
http://aleph0.clarku.edu/~djoyce/java/elements/bookVII/propVII2.html
If EF is the greatest common measure of segments AB and CD and GH is a common measure of AB and CD, then there exists a natural number n such that $\frac{EF}{GH} = n$.

If a segment AB is longer than a segment CD and these segments are commensurable with a segment EF, then $\frac{AB}{EF} > \frac{CD}{EF}$.

2. Incommensurable segments. There exist segments that are not commensurable. For example, a side and diagonal of a square are not commensurable, see, for example,
http://www.learner.org/courses/mathilluminated/units/3/textbook/03.php
Segments that are not commensurable are called *incommensurable*.

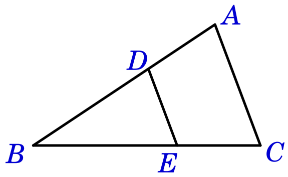
For incommensurable segments AB and CD the ratio $\frac{AB}{CD}$ is defined as the unique real number r such that

- $r < \frac{EF}{CD}$ for any segment EF, which is longer than AB and commensurable with CD;
- $\frac{EF}{CD} < r$ for any segment EF, which is shorter than AB and commensurable with CD.

3. Thales' Theorem. (See Sections 113 and 114 of the textbook.) *Let ABC be a triangle, D be a point on AB and E be a point on BC. If $DE \parallel AC$, then*

$$\frac{BD}{DA} = \frac{BE}{EC}.$$

□



Corollary. *Under the assumptions of Thales' Theorem,*

$$\frac{BD}{BA} = \frac{BE}{BC} = \frac{DE}{AC}.$$

□

4. Distance. If we choose a segment AB and call it the unit, then we can assign to any other segment CD the number $\dfrac{CD}{AB}$, call it the *length* of CD and denote by $|CD|$.

Further, the length $|CD|$ of segment CD is called then the *distance* between points C and D and denote by $\operatorname{dist}(C,D)$. Of course, $\operatorname{dist}(C,D)$ depends on the choice of AB. Define $|CD|$ and $\operatorname{dist}(C,D)$ to be 0 if $C = D$.

The distance between points has the following properties:

- it is symmetric, $\operatorname{dist}(C,D) = \operatorname{dist}(D,C)$ for any points C, D;
- $\operatorname{dist}(C,D) = 0$ if and only if $C = D$;
- triangle inequality, $\operatorname{dist}(C,D) \le \operatorname{dist}(C,E) + \operatorname{dist}(E,D)$.

The first two of these properties are obvious, the last one was proven, see Section 26 of the textbook.

5. Metric spaces. In mathematics there are many functions which have these 3 properties. Therefore it was productive to create the following notion of metric space. A *metric space* is an arbitrary set X equipped with a function $d : X \times X \to \mathbb{R}_+$ such that

- $d(a,b) = d(b,a)$ for any $a,b \in X$;
- $d(a,b) = 0$ if and only if $a = b$;
- $d(a,b) \le d(a,c) + d(c,b)$ for any $a,b,c \in X$.

Thus the plane with a selected unit segment AB and $d(C,D) = \operatorname{dist}(C,D)$ (as in Section 4 above) is a metric space.

6. Definition of similarity transformations. Let X and Y be metric spaces with distances d_X and d_Y, respectively. A map $T : X \to Y$ is said to be a *similarity transformation* with *ratio* $k \in \mathbb{R}$, $k \ge 0$, if $d_Y(T(a),T(b)) = kd_X(a,b)$ for any $a,b \in X$.

Other terms used in the same situation: a similarity transformation may call a *dilation*, or *dilatation*, the ratio may call also the *coefficient* of the dilation.

Most often the notion of similarity transformation is applied when $X = Y$ and $d_X = d_Y$. We will consider it when $X = Y$ is the plane or the 3-space and the distance is defined via the length of the corresponding segment, and the length is defined by a choice of unit segment, as above.

General properties of similarity transformations.

1. Any isometry is a similarity transformation with ratio 1.

2. Composition $S \circ T$ of similarity transformations T and S with ratios k and l, respectively, is a similarity transformation with ratio kl.

7. Homothety. A profound example of dilation with ratio different from 1 is a homothety.

Definition. Let k be a positive real number, O be a point on the plane. The map which maps O to itself and any point $A \neq O$ to a point B such that the rays OA and OB coincide and $\dfrac{OB}{OA} = k$ is called the *homothety* centered at O with ratio k.

Composition $T \circ S$ of homotheties T and S with the same center and ratios k and l, respectively, is the homothety with the same center and the ratio kl. In particular, any homothety is invertible and the inverse transformation is the homothety with the same center and the inverse ratio.

Theorem 1. *A homothety T with ratio k is a similarity transformation with ratio k.*

Proof. We need to prove that $\frac{T(A)T(B)}{AB} = k$ for any segment AB. Consider, first, the case when O does not belong to the line AB. Then OAB is a triangle, and $OT(A)T(B)$ is also a triangle.

Assume that $k < 1$. Then $T(A)$ belongs to the segment OA. Draw a segment $T(A)C$ parallel to AB with the end point C belonging to OB. Then by Corollary of Thales' Theorem, $\frac{OC}{OB} = \frac{OT(A)}{OA} = k$. Therefore $C = T(B)$. Again, by Corollary of Thales' Theorem, $\frac{T(A)T(B)}{AB} = \frac{OT(A)}{OA} = k$.

If $k > 1$, then A belongs to $OT(A)$. Draw the segment AC parallel to $T(A)T(B)$ and having the end point C on the segment $OT(B)$. By Corollary of Thales' Theorem, $\frac{OT(B)}{OC} = \frac{OT(A)}{OA} = k$. Therefore $C = T(B)$. Again, by Corollary of Thales' Theorem, $\frac{T(A)T(B)}{AB} = \frac{OT(A)}{OA} = k$.

The easy case, when points A, B, O are collinear, consider as an exercise. □

Corollary. *Any similarity transformation T with ratio k of the plane is a composition of an isometry and a homothety with ratio k.*

Proof. Consider a composition $T \circ H$ of T with a homothety with ratio k^{-1}. This composition is a similarity transformation with ratio $k^{-1}k = 1$, that is an isometry. Denote this isometry by I. Thus $I = T \circ H$. Multiply both sides of this equality by H^{-1} from the right hand side: $I \circ H^{-1} = T \circ H \circ H^{-1} = T$. □

Theorem 2. *A similarity transformation of a plane is invertible.*

Proof. By Corollary of Theorem 1, any similarity transformation T is a composition of an isometry and a homothety. A homothety is invertible, as was noticed above. An isometry of the plane is a composition of at most three reflections. Each reflection is invertible, because its composition with itself is the identity. A composition of invertible maps is invertible. □

Corollary. *The transformation inverse to a similarity transformation T with ratio k is a similarity transformation with ratio k^{-1}.*

8. Similar figures. Plane figures F_1 and F_2 are said to be *similar* if there exists a similarity transformation T such that $T(F_1) = F_2$.

Any two congruent figures are similar. In particular, any two lines are congruent and hence similar, any two rays are congruent and hence similar.

Segments are not necessarily congruent, but nonetheless any two segments are similar. Indeed, first, by a congruence transformation one can make any segment parallel to another segment, and then find a homothety mapping one of the segments to the other one.

Any two circles (or any two disks) are similar. Indeed, if the circles have the same radius, then one can find a translation mapping one them onto the other one, otherwise one can find a homothety mapping one of them onto the other one.

Theorem 3. *A figure similar to a segment is a segment.*

Lemma. Characterization of points belonging to a segment. *A point C belongs to a segment AB if and only if $|AC| + |CB| = |AB|$.*

Proof. If C belongs to AB, then the segment AB is the sum of segments AC and CB and hence $|AC| + |CB| = |AB|$.

If point C does not belong to the line AB, then $|AC| + |CB| > |AB|$ by Theorem 26 of the textbook (the triangle inequality).

If C belongs to the line AB, but does not belong to the segment AB, then either AC contains B or BC contains A. In the former case $|AC| > |AB|$, in the latter $|BC| > |AB|$. In both cases, the equality $|AC| + |CB| = |AB|$ does not hold true. □

Proof of Theorem 3. The relation $|AC| + |CB| = |AB|$ characterizing the set of points of segment AB is invariant under a similarity transformation. Indeed, if T is a similarity transformation with ratio k, then $|T(A)T(C)| = k|AC|$, $|T(C)T(B)| = k|CB|$, and $|T(A)T(B)| = k|AB|$. Therefore, if C belongs to AB, then $|AC| + |CB| = |AB|$,

$$\begin{aligned}|T(A)T(C)| + |T(C)T(B)| &= k|AC| + k|CB| \\ &= k(|AC| + |CB|) = k|AB| = |T(A)T(B)|\end{aligned}$$

and hence $T(C)$ belongs to the segment $T(A)T(B)$. Thus the image of segment $[AB]$ under T is contained in the segment $[T(A)T(B)]$:

$$T([AB]) \subset [T(A)T(B)].$$

Similarly, $T^{-1}[T(A)T(B)] \subset [AB]$. Therefore,

$$[T(A)T(B)] = TT^{-1}[T(A)T(B)] \subset T[AB].$$

Hence $T([AB]) = [T(A)T(B)]$. □

Exercises.
1. Prove that a figure similar to a line is a line.
2. Prove that a figure similar to a ray is a ray.
3. Prove that a figure similar to a circle is a circle.

Theorem 4. *A figure similar to an angle is an angle. Two angles are similar if and only they are congruent.*

Proof. Recall that an angle is a figure consisting of two rays starting from the same point. Since a figure similar to a ray is a ray and a similarity transformation of an angle onto another figure should map the common point of the rays to a common point of their images, the image of an angle under a similarity transformation is an angle.

Consider now two similar angles and prove that they are congruent. Let T be a similarity mapping an angle $\angle A$ to an angle $\angle B$. Let the ratio of T be k. Let H be the homothety centered at the vertex of B with ratio k^{-1}. The image of $\angle B$ under H is $\angle B$. Therefore the composition $H \circ T$ maps $\angle A$ onto $\angle B$. This composition is a similarity mapping with ratio $k \cdot k^{-1} = 1$. Hence, $H \circ T$ is an isometry mapping $\angle A$ onto $\angle B$. □

9. Similarity tests for triangles.

Theorem 5 (AA-test)***.*** *If in triangles ABC and $A'B'C'$ the angles $\angle A$, $\angle A'$ are congruent and angles $\angle B$, $\angle B'$ are congruent, then $\triangle ABC$ is similar to $\triangle A'B'C'$.*

Proof. Without loss of generality we may assume that $A'B'$ is shorter than AB. Find a point D on AB such that $|BD| = |B'A'|$. Draw a segment DE parallel to AC. By ASA test for congruence of triangles, $\triangle A'B'C'$ is congruent to $\triangle DBE$. By Corollary of Thales' Theorem, $\frac{DB}{AB} = \frac{BE}{BC}$. Hence, the homothety centered at B with ratio $\frac{DB}{AB}$ maps $\triangle ABC$ onto $\triangle DBE$. □

Theorem 6 (SAS-test)**.** *If in triangles ABC and $A'B'C'$ the angles $\angle A$, $\angle A'$ are congruent and*

$$\frac{A'B'}{AB} = \frac{A'C'}{AC}$$

then $\triangle ABC$ is similar to $\triangle A'B'C'$.

Proof. Without loss of generality we may assume that $A'B'$ is shorter than AB. Find a point D on AB such that $|BD| = |B'A'|$. Draw a segment DE parallel to AC. By Corollary of Thales' Theorem, $\frac{DB}{AB} = \frac{BE}{BC}$, and therefore $|BE| = |B'C'|$. By SAS test for congruence of triangles, $\triangle A'B'C'$ is congruent to $\triangle DBE$. The homothety centered at B with ratio $\frac{DB}{AB}$ maps $\triangle ABC$ onto $\triangle DBE$. □

Theorem 7 (SSS-test)**.** *If in triangles ABC and $A'B'C'$*

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA}$$

then $\triangle ABC$ is similar to $\triangle A'B'C'$.

Proof. Without loss of generality we may assume that $A'B'$ is shorter than AB. Find a point D on AB such that $|BD| = |B'A'|$. Draw a segment DE parallel to AC. By Corollary of Thales' Theorem, $\frac{DB}{AB} = \frac{BE}{BC} = \frac{DE}{AC}$. Therefore $|BE| = |B'C'|$ and $|DE| = |A'C'|$. By SSS test for congruence of triangles, $\triangle A'B'C'$ is congruent to $\triangle DBE$. The homothety centered at B with ratio $\frac{DB}{AB}$ maps $\triangle ABC$ onto $\triangle DBE$. □