Isometries.

Congruence mappings as isometries. The notion of *isometry* is a general notion commonly accepted in mathematics. It means a mapping which preserves distances. The word *metric* is a synonym to the word *distance*. In the context of this course, an isometry is a mapping of the plane that maps each segment s to a segment s' congruent to s. Therefore each congruence mapping is an isometry. In fact, each isometry of the plane is a congruence mapping.

Here we study isometries of the plane.

Exercise. Generalize everything into the setup of the 3-space.

Recovering an isometry from images of three points.

Theorem 1. *An isometry of the plane can be recovered from its restriction to any triple of non-collinear points.*

Proof. Given images A', B' and C' of non-collinear points A, B, C under and isometry, let us find the image of an arbitrary point X. Using a compass, draw circles c_A and c_B centered at A' and B' of radii congruent to AX and BX, respectively. They intersect in at least one point, because segments AB and $A'B'$ are congruent and the circles centered at A and B with the same radii intersect at X. There may be two intersection point. The image of X must be one of them. In order to choose the right one, measure the distance between C and S and choose the intersection point X' of the circles c_A and c_B such that $C'X'$ is congruent to CX. □

In fact, there are exactly two isometries with the same restriction to a pair of distinct points. They can be obtained from each other by composing with the reflection about the line connecting these points.

Isometries as compositions of reflections.

Theorem 2. *Any isometry of the plane is a composition of at most three reflections.*

Proof. Choose three non-collinear points A, B, C. By theorem *1*, it would suffice to find a composition of at most three reflections which maps A, B and C to their images under a given isometry S.

First, find a reflection R_1 which maps A to $S(A)$. The axis of such a reflection is a perpendicular bisector of the segment $AS(A)$. It is uniquely defined, unless $S(A) = A$. If $S(A) = A$, one can take either a reflection about any line passing through A, or take, instead of reflection, an identity map for R_1 .

Second, find a reflection R_2 which maps segment $S(A)R_1(B)$ to $S(A)S(B)$. The axis of such a reflection is the bisector of angle $\angle R_1(B)S(A)S(B)$.

The reflection R_2 maps $R_1(B)$ to $S(B)$. Indeed, the segment $S(A)R_1(B) = R_1(AB)$ is congruent to AB (because R_1 is an isometry), AB is congruent to $S(A)S(B) = S(AB)$ (because S is an isometry), therefore $S(A)R_1(B)$ is congruent to $S(A)S(B)$. Reflection R_2 maps the ray $S(A)R_1(B)$ to the ray $S(A)S(B)$, preserving the point $S(A)$ and distances. Therefore it maps $R_1(B)$ to $S(B)$.

Triangles $R_2 \circ R_1(\triangle ABC)$ and $S(\triangle ABC)$ are congruent via an isometry $S \circ (R_2 \circ R_1)^{-1} = S \circ R_1 \circ R_2$, and the isometry is identity on the side $S(AB) = R_2 \circ R_1(AB)$. Now either $R_2(R_1(C)) = C$ and then $S = R_2 \circ R_1$, or the triangles $R_2 \circ R_1(\triangle ABC)$ and $S(\triangle ABC)$ are symmetric about their common side $S(AB)$. In the former case $S = R_2 \circ R_1$, in the latter case denote by R_3 the reflection about $S(AB)$ and observe that $S = R_3 \circ R_2 \circ R_1$. □

Translations and central symmetries. A map of the plane to itself is called a *translation* if, for some fixed points A and B, it maps a point X to a point $T(X)$ such that $XT(X)BA$ is a parallelogram.

Here we have to be careful with the notion of parallelogram, because a parallelogram may degenerate to a figure in a line. Not any degenerate quadrilateral fitting in a line deserves to be called a parallelogram, although any two sides of such a degenerate quadrilateral are parallel. By a parallelogram we mean a sequence of four segments KL, LM, MN and MK such that KL is congruent and parallel to MN and LM is congruent and parallel to MK. This definition describes the usual parallelograms, for which congruence can be deduced from parallelness and vice versa, and the degenerate parallelograms.

Theorem 3. *For any points A and B there exists a translation mapping A to B. A translation is an isometry.*

Proof. Any point A, B and X can be completed in a unique way to a parallelogram $ABXY$. Define $T(X) = Y$. For any points X, Y the quadrilateral $XYT(Y)T(X)$ is a parallelogram. Therefore, T is an isometry. □

Denote by T_{AB} the translation which maps A to B.

Theorem 4. *The composition of any two translations is a translation.*

Theorem *4* means that $T_{BC} \circ T_{AB} = T_{AC}$.

Fix a point O. A map of the plane to itself which maps a point A to a point B such that O is a midpoint of the segment AB is called the *symmetry about a point* O.

Theorem 5. *A symmetry about a point is an isometry.*

Proof. SAS-test for congruent triangles (extended appropriately to degenerate triangles.) □

Theorem 6. *The composition of any two symmetries in a point is a translation. In details,* $S_B \circ S_A = T_{\frac{1}{2}\overrightarrow{AB}}$*, where* S_X *denotes the symmetry about point* X*.*

Remark. The equality

$$S_B \circ S_A = T_{\frac{1}{2}\overrightarrow{AB}}$$

implies a couple of other useful equalities. Namely, compose both sides of this equality with S_B from the left:

$$S_B \circ S_B \circ S_A = S_B \circ T_{\frac{1}{2}\overrightarrow{AB}}$$

Since $S_B \circ S_B$ is the identity, it can be rewritten as

$$S_A = S_B \circ T_{\frac{1}{2}\overrightarrow{AB}}.$$

Similarly, but multiplying by S_A from right, we get

$$S_B = T_{\frac{1}{2}\overrightarrow{AB}} \circ S_A.$$

Corollary. *The composition of an even number of symmetries in points is a translation; the composition of an odd number of symmetries in points is a symmetry in a point.*

Compositions of two reflections.

Theorem 7. *The composition of two reflections in non-parallel lines is a rotation about the intersection point of the lines by the angle equal to doubled angle between the lines. In formula:*

$$R_{AC} \circ R_{AB} = Rot_{A,2\angle BAC},$$

where R_{XY} *denotes the reflection in line* XY*, and* $Rot_{X,\alpha}$ *denotes the rotation about point* X *by angle* α*.*

Theorem 8. *The composition of two reflections in parallel lines is a translation in a direction perpendicular to the lines by a distance twice larger than the distance between the lines.*

More precisely, if lines AB *and* CD *are parallel, and the line* AC *is perpendicular to the lines* AB *and* CD*, then*

$$R_{CD} \circ R_{AB} = T_{2\overrightarrow{AC}}.$$

Application: finding triangles with minimal perimeters. We have considered the following problem:

Problem 1. *Given a line l and points A, B on the same side of l, find a point $C \in l$ such that the broken line ACB would be the shortest.*

Recall that a solution of this problem relies on reflection. Namely, let $B' = R_l(B)$. Then the desired C is the intersection point of l and AB'.

Notice that this problem can be reformulated as finding $C \in l$ such that the perimeter of the triangle ABC is minimal.

Problem 2. *Given lines l, m and a point A, find points $B \in l$ and $C \in m$ such that the perimeter of the triangle ABC is minimal.*

Construction that solves Problem 2. Reflect point A in l and m, that is find $B' = R_l(A)$ and $C' = R_m(A)$. Then $B = l \cap B'C'$ and $C = m \cap B'C'$. Exercise: provide a proof and research.

Problem 3. *Given lines l, m and n, no two of which are parallel to each other. Find points $A \in l$, $B \in m$ and $C \in n$ such that triangle ABC has minimal perimeter.*

If we knew a point $A \in l$, the problem would be solved as Problem 2: we would connect points $R_m(A)$ and $R_n(A)$ and take for B and C the intersection points of this line with m and n. So, we have to find a point $A \in l$ such that the segment $R_m(A)R_n(A)$ would be minimal.

The end points $R_m(A)$, $R_n(A)$ of this segment belong to the lines $R_m(l)$ and $R_n(l)$ and are obtained from the same point $A \in l$. Therefore

$$R_n(A) = R_n(R_m(R_m(A))) = R_n \circ R_m(B),$$

where $B \in R_m(l)$. So, one end point is obtained from another by $R_n \circ R_m$.

By Theorem *8*, $R_n \circ R_m$ is a rotation about the point $m \cap n$. We look for a point B on $R_m(l)$ such that the segment $BR_n \circ R_m(B)$ is minimal.

The closer a point to the center of rotation, the closer this point to its image under the rotation. Therefore the desired B is the base of the perpendicular dropped from $m \cap n$ to $R_m(l)$. Hence, the desired A is the base of perpendicular dropped from $m \cap n$ to l.

Since all three lines are involved in the conditions of the problem in the same way, the desired points B and C are also the end points of altitudes of the triangle formed by lines l, m, n.

Composition of rotations.

Theorem 9. *The composition of rotations (about points which may be different) is either a rotation or translation.*

Prove this theorem by representing each rotation as a composition of two reflections about a line. Choose the lines in such a way that the second line in the representation of the first rotation would coincide with the first line in the representation of the second rotation. Then in the representation of the composition of two rotations as a composition of four reflections the two middle reflections would cancel and the whole composition would be represented as a composition of two reflections. The angle between the axes of these reflections would be the sum of of the angles in the decompositions of the original rotations. If this angle is zero, and the lines are parallel, then the composition of rotations is a translation by Theorem *8*. If the angle is not zero, the axes intersect, then the composition of the rotations is a rotations around the intersection point by the angle which is the sum of angles of the original rotations.

Similar tricks with reflections allows to simplify other compositions.

Glide reflections. A reflection about a line l followed by a translation along l is called a *glide reflection*. In this definition, the order of reflection and translation does not matter, because they commute: $R_l \circ T_{AB} = T_{AB} \circ R_l$ if $l \parallel AB$.

Theorem 10. *The composition of a central symmetry and a reflection is a glide reflection.*

Use the same tricks as for Theorem *9*

Classification of plane isometries.

Theorem 11. *Any isometry of the plane is either a reflection about a line, or rotation, or translation, or gliding reflection.*

This theorem can be deduced from Theorem *2* by taking into account relations between reflections in lines. By Theorem *2*, any isometry of the plane is a composition of at most 3 reflections about lines. By Theorems *7* and *8*, a composition of two reflections is either rotation about a point or translation.

Lemma. A composition of three reflections is either a reflection, or a gliding reflection.

Proof. If all three axes of the reflections are parallel, then the firs two can be translated without changing of their composition (the composition of reflections about two parallel lines depends only on the direction of lines and the distance between them). By translating the first two lines, make the second of them coinciding with the third line. Then in the total composition they cancel, and the composition is just the reflection in the first line.

If not all three lines are parallel, then the second is not parallel to one of the rest. The composition of reflections about these two non-parallel lines is a rotation, and the lines can be rotated simultaneously about their intersection point by the same angle without changing of the composition.

By an appropriate rotation, make the middle line perpendicular to the line which was not rotated. Then by rotating of these two perpendicular lines about their intersection point, make the middle one parallel to the other line. Now the configuration of lines consists of two parallel lines and a line perpendicular to them. The composition of reflections about them (the order does not matter any more, because they commute) is a gliding symmetry. □