# 1. Equations for lines on the plane and planes in the space.

## 1.1. General implicit vector equation.

$$\mathbf{a} \cdot \mathbf{r} = \alpha \tag{1}$$

This equation defines a line in the plane and a plane in the 3-space. Here $\mathbf{r}$ is the radius-vector of a variable point; $\mathbf{a}$ and $\alpha$ are constants, coefficients of the equation (1). The geometric meaning of vector $\mathbf{a}$ is that it is orthogonal to the line/plane defined by the equation.

The equation imposes a restriction on this point:

***Theorem 1.*** *If $\mathbf{a} \neq 0$, then the radius-vector of a point satisfies this equation iff the point belongs to a line or a plane, depending the environment (line on the plane, or plane in the space).*

Prove this theorem. The main idea behind the proof is that the right hand side of (1) is proportional to the projection of $\mathbf{r}$ to the line determined by $\mathbf{a}$, and the equation says that this projection is fixed.

***Question.*** What figure is defined by (1) if $\mathbf{a} = 0$?

**1.2. General implicit equation in coordinates.** On the plane with Cartesian coordinates $x$, $y$ the equation (1) can be re-written as

$$Ax + By + C = 0 \tag{2}$$

To relate this to (1), put $\mathbf{a} = (A, B)$ and $\alpha = -C$.

In the 3-space, (1) can be rewritten as

$$Ax + By + Cz + D = 0 \tag{3}$$

To relate this to (1), put $\mathbf{a} = (A, B, C)$ and $\alpha = -D$.

**1.3. Normal equations.** If in equation (1) the coefficient $\mathbf{a}$ is a unit vector (that is $|\mathbf{a}| = 1$) and $\alpha \geq 0$, then it is called a *normal equation*.

***Theorem 2.*** *Let*

$$\mathbf{n} \cdot \mathbf{r} = p \tag{4}$$

*be a normal equation. Then $p$ is the distance from the origin to the line/plane defined by (4). More generally, $|\mathbf{n} \cdot \mathbf{r}_0 - p|$ is the distance from the point with radius-vector $\mathbf{r}_0$ to the line/plane defined by (4).*

To obtain a normal equation from equation (1), divide both sides of (1) by $\pm|\mathbf{a}|$. The sign should be chosen in order to make the right hand side non-negative.

***Question.*** How many normal equations may a line/plane have?

**1.4. Passing through a point.**

$$\mathbf{a}\cdot(\mathbf{r}-\mathbf{r}_0)=0 \tag{5}$$

defines a line/plane passing through the point with radius-vector $\mathbf{r}_0$.

In a Cartesian coordinate system on the plane this equation looks like that:

$$A(x-x_0)+B(y-y_0)=0. \tag{6}$$

Here $(A,B)=\mathbf{a}$ is a vector orthogonal to the line, $(x_0,y_0)$ - the coordinates of the point through which the line passes.

In a Cartesian coordinate system in the 3-space this equation looks like that:

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0. \tag{7}$$

Here $(A,B,C)=\mathbf{a}$ is a vector orthogonal to the plane, $(x_0,y_0,z_0)$ - the coordinates of the point through which the plane passes.

**1.5. Intercepts equations.**

***Theorem 3.*** *Let $a,b,c$ be real numbers, $abc\neq 0$ (i.e., none of these numbers is zero). The equation*

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \tag{8}$$

*defines in the 3-space a plane meeting the coordinate axes at points $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$, respectively.*

*Similarly, the equation*

$$\frac{x}{a}+\frac{y}{b}=1 \tag{9}$$

*defines on the plane a line meeting the coordinate axes at points $(a,0)$ and $(0,b)$, respectively.*

***Exercises.*** **1.** Prove this theorem.
**2.** Find similar equations for planes parallel to some of the axes.

## 2. Equations for line.

**2.1. Parametric equations.** If $\mathbf{a}$ is a non-zero vector, then the equation

$$\mathbf{r}=\mathbf{a}t+\mathbf{r}_0 \tag{10}$$

defines a line passing through the point with radius-vector $\mathbf{r}_0$ and parallel to vector $\mathbf{a}$ in the following sense:

***Theorem 1.*** *The image of the line $\mathbb{R}$ under the mapping*

$$t \mapsto \mathbf{a}t + \mathbf{r}_0$$

*is the line passing through the point with radius-vector $\mathbf{r}_0$ and parallel to vector $\mathbf{a}$.*

The number $t$ is called a *parameter*. It should be thought of as an address of the point with the radius-vector $\mathbf{r} = \mathbf{a}t + \mathbf{r}_0$, while the formula (10) is a way to find this point by its address $t$.

Here the space under consideration is not mentioned, because it does not matter: both on the plane and in the 3-space the same happens.

In the plane the parametric equation $\mathbf{r} = \mathbf{a}t + \mathbf{r}_0$ can be re-written in coordinates as follows

$$\begin{cases} x = & pt + x_0 \\ y = & qt + y_0 \end{cases} \tag{11}$$

if $\mathbf{a} = (p, q)$ and $\mathbf{r}_0 = (x_0, y_0)$. Similarly, one can re-write the parametric equation for a line in the 3-space.

### 2.2. An implicit equation of a line parallel to a vector.

***Theorem 2.*** *On the plane a line passing through a point $(x_0, y_0)$ and parallel to vector $(p, q)$ with $pq \neq 0$ is defined by equation*

$$\frac{x - x_0}{p} = \frac{y - y_0}{q} \tag{12}$$

***Exercises.*** **1.** Prove this theorem.
(Hint: The meaning of the equation (12) is that vectors $(x - x_0, y - y_0)$ and $(p, q)$ have proportional coordinates. What is a geometric meaning of this proportionality?)
**2.** Find similar equations for lines parallel to one of the axes.
**3.** Find similar equations for lines in the 3-space.

**2.3. Relation between parametric and implicit equations.** A line given by equation

$$\frac{x - x_0}{p} = \frac{y - y_0}{q}$$

can be defined by a parametric equation using the following trick: put

$$t = \frac{x - x_0}{p} = \frac{y - y_0}{q}$$

then

$$\begin{cases} x = & pt + x_0 \\ y = & qt + y_0 \end{cases}$$

### 2.4. Line passing through given points.

***Theorem 3.*** *On the plane a line passing through points* $(x_1, y_1)$ *and* $(x_2, y_2)$ *with* $(x_1 - x_2)(y_1 - y_2) \neq 0$ *is defined by equation*

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}. \tag{13}$$

***Exercises.*** **1.** Prove this theorem.
**2.** What is the geometric meaning of the assumption $(x_1-x_2)(y_1-y_2) \neq 0$?
**3.** What is the role of the assumption $(x_1 - x_2)(y_1 - y_2) \neq 0$?
**4.** Is there a way of understanding equation (13) which would allow to consider points with $(x_1 - x_2)(y_1 - y_2) = 0$?
**5.** Find similar equations for lines in the 3-space.

### 2.5. Implicit vector equation for line in the space.

***Theorem 4.*** *Let* $\mathbf{a} \neq 0$, $\mathbf{b}$ *be vectors perpendicular to each other. Then the equation*

$$\mathbf{a} \times \mathbf{r} = \mathbf{b} \tag{14}$$

*defines a line parallel to vector* $\mathbf{a}$ *and contained in the plane passing through the origin and perpendicular to* $\mathbf{b}$.

Prove this theorem. The main idea behind the proof is that the equation (14) means that vectors $\mathbf{r}$ and $\mathbf{a}$ determine a parallelogram contained in the plane perpendicular to $\mathbf{b}$ and having a fixed area.
***Questions.*** **1.** What if the vectors $\mathbf{a}$ and $\mathbf{b}$ are not perpendicular?
**2.** What if $\mathbf{a} = 0$, what figure is defined by equation (14) then?
**3.** Can any line in the 3-space be defined by an equation of this form?

### 2.6. Line passing through a point.

***Theorem 5.*** *Let* $\mathbf{a}$ *be a non-zero vector. A line passing through the point with radius-vector* $\mathbf{r}_0$ *and parallel to* $\mathbf{a}$ *can be defined by equation*

$$\mathbf{a} \times (\mathbf{r} - \mathbf{r}_0) = 0 \tag{15}$$

***Theorem 6.*** *the line passing through the points with radius-vectors* $\mathbf{r}_1$ *and* $\mathbf{r}_2$ *is defined by equation*

$$(\mathbf{r}_1 - \mathbf{r}_2) \times (\mathbf{r} - \mathbf{r}_1) = 0. \tag{16}$$

Prove theorems *5* and *6*.
Hint: two vectors are parallel iff their skew product equals zero.

**2.7. Normal vector implicit equation for a line.** If in equation (14) the coefficient $\mathbf{a}$ is a unit vector (that is $|\mathbf{a}| = 1$), then (14) is called a *normal equation*.

***Theorem 7.*** *Let*

$$\mathbf{n} \times \mathbf{r} = \mathbf{p} \tag{17}$$

*be a normal equation. Then* $|\mathbf{p}|$ *is the distance from the origin to the line defined by* (17). *More generally,* $|\mathbf{n} \times \mathbf{r}_0 - \mathbf{b}|$ *is the distance from the point with radius-vector* $\mathbf{r}_0$ *to the line defined by* (17).

To obtain a normal equation from equation (14), divide both sides of (14) by $\pm|\mathbf{a}|$.

***Question.*** How many normal equations may a line in the 3-space have?