SUPPLEMENTARY MATERIALS TO MAT 513 ANALYSIS FOR TEACHERS SPRING 2015

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1. Linearly Ordered Sets

1.1. **Definition.** A set X with binary relation $\prec$ is called a *totally* or *ordered set* and the relation $\prec$ is called a *total* or *linear order* in X if

- $\prec$ is *transitive*,
  that is for any $a, b, c \in X$, $a \prec b$ and $b \prec c$ imply $a \prec c$;
- for any $a, b \in X$ either $a \prec b$, or $b \prec a$, or $a = b$.

The second condition is called *trichotomy*: for any two element a, b of X exactly one of *three* statements is true: either $a \prec b$, or $b \prec a$, or $a = b$.
Extra notation / definitions: In a totally ordered set $(X, \prec)$, the relation $\prec$ defines (and is defined by) relations

- $\succ$: $a \succ b$ means that $b \prec a$;
- $\preceq$: $a \preceq b$ means that $a \prec b$ or $a = b$;
- $\succeq$: $a \succeq b$ means that $a \succ b$ or $a = b$.

In a totally ordered set $(X, \prec)$, the relation $\succ$ is also a total order. It is called *opposite* or *dual* to $\prec$.

There are many total orders of quite different origins.

Examples.
1. Numbers with the usual order. The set X here may be the set of all natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$, the set of all positive rational numbers $\mathbb{Q}_{>0}$, or the set of all rational numbers $\mathbb{Q}$, or any subset of these sets.
2. The set of letters $\{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$ with the alphabetical order.
3. The set of all English words with the lexicographical order.

1.2. **Monotone maps.** Let $(X, \prec)$ and $(Y, \prec)$ be linearly ordered sets, a map $f : X \to Y$ is called *increasing* or *monotone increasing* if

$$\forall a, b \in X : \; a \prec b \implies f(a) \prec f(b).$$

A map $f : X \to Y$ is called *decreasing* or *monotone decreasing* if

$$\forall a, b \in X : \; a \prec b \implies f(a) \succ f(b).$$

Exercise 1. Prove that *a composition of monotone maps is monotone.*

The Exercise may be formulated in more details. For example, here is its special case: if $(X, \prec)$, $(Y, \prec)$ and $(Z, \prec)$ are linearly ordered sets, $f : X \to Y$ is increasing and $g : Y \to Z$ is decreasing, then $g \circ f : X \to Z$ is decreasing.

1.3. **Making new orders from old ones.** For any linearly ordered set $(X, \prec)$ and a subset Y of X, the order relation $\prec$ in X defines the relation in Y, making Y linearly ordered. Namely for $a, b \in Y$ we write $a \prec_Y b$ if $a \prec b$. Notice that $\prec_Y$ is a linear order in Y: it is transitive and satisfies the trichotomy. The order $\prec_Y$ is said to be *induced* in Y. It is essentially coincides with $\prec$ and is denoted usually by the same symbol.

1.4. **On notation.** In general definition of ordered set above we used an unusual symbol $\prec$ in order to emphasize that an order may have nothing to do with the usual order of numbers. Symbols may vary. On the other hand, our interest to totally ordered sets is motivated by the standard ordering of numbers. Therefore in what follows we will denote an order under consideration by symbol $<$. Respectively, instead of symbol $\preceq$ we will use $\leq$, instead of $\succ$ we will use $>$, and instead of $\succeq$ we will use $\geq$.

We will often speak about a linearly ordered set X without introducing the symbol by which we denote the order. In such a case the order is denoted by $<$.

1.5. **Intervals and rays.** Let $(X, <)$ be a totally ordered set, and $a, b \in X$. The set $\{x \in X \mid a < x, x < b\}$ is called the *open interval* with endpoints a, b and is denoted by (a, b).

The set $\{x \in X \mid a \leq x \text{ and } x \leq b\}$ is called the *closed interval* with endpoints a, b and is denoted by $[a, b]$.

Sets $\{x \in X \mid a < x\}$ and $\{x \in X \mid x < a\}$ are called *open rays* with endpoint a and are denoted by (a, ∞) and $(-\infty, a)$, respectively.

Here the symbol ∞ does not denote any element of X.

The sets $\{x \in X \mid a \leq x\}$ and $\{x \in X \mid x \leq a\}$ are called *closed ray* with endpoint a and denoted by $[a, \infty)$ and $(-\infty, a]$.

Intervals are related with rays: $(a, b) = (-\infty, b) \cup (a, \infty)$ and $[a, b] = (-\infty, b] \cup [a, \infty)$. There are also half-open half closed intervals $(a, b]$ and $[a, b)$ which can be introduced by formulas $[a, b) = (-\infty, b) \cup [a, \infty)$ and $(a, b] = (-\infty, b] \cup (a, \infty)$. Sometimes rays are also called intervals.

Theorem 1. *Any open ray is a union of closed rays: namely,* $(a, \infty) = \cup_{x>a}[x, \infty)$ *and* $(-\infty, a) = \cup_{x<a}(-\infty, x]$ *for any* $a \in X$.

Problem 1. Is a ray $(2, \infty) \subset \mathbb{Q}$ representable as a union of closed rays in $\mathbb{Q}$?

Problem 2. Is a ray $(2, \infty) \subset \mathbb{N}$ representable as a union of closed rays in $\mathbb{N}$?

1.6. **Greatest and least.** Let X be a linearly ordered set, $A \subset X$. An element $b \in A$ is called the *greatest* in A if $b \geq x$ for any $x \in A$. An element $b \in A$ is called the *least* in A if $b \leq x$ for any $x \in A$.

Control question. What would happen to the definition of greatest element above if one would replace the symbol $\geq$ by $>$? Would the defined notion stay the same?

Exercise 2. Prove that in any set there exists at most one greatest element and at most one least element.

Exercise 3. What is the greatest element in $[a, b]$?

Exercise 4. Is there a greatest element in (a, b)? Does the answer depend on the ordered set?

1.7. **Bounds.** Let $(X, <)$ be a totally ordered set and $A \subset X$. Then

- $u \in X$ is called an *upper bound* for A if $a \leq u$ for any $a \in A$;
- $l \in X$ is called a *lower bound* for A if $l \leq a$ for any $a \in A$.

Control question. Is the greatest element g of a set A an upper bound for A?

A set A is called

- *bounded from above* if it has an upper bound,
- *bounded from below* if it has a lower bound;
- *bounded* if it has both upper and lower bounds.

Problem 3. Let X be a linearly ordered set, and $A \subset Y \subset X$.
Can it happen that A is bounded in X, but not in Y?
Can it happen that A is bounded in Y, but not in X?

Denote by $U(A)$ the set of all upper bounds of A and by $L(A)$ the set of lower bounds of A.

The least element of $U(A)$ is called the *supremum* of A or the *least upper bound* of A. It is denoted by $\sup A$.

Notice that each set has at most one supremum.

The greatest element of $L(A)$ is called the *infimum* of A or the *greatest lower bound* of A. It is denoted by $\inf A$.

Notice that each set has at most one infimum.

Problem 4. Prove that if a set has the greatest element, then this element is the supremum of that set. Dually, if a set has the least element, then this element is the infimum of that set.

Examples. In $\mathbb{Q}$:

- $\inf\{0, 1, 2\} = 0$, $\sup\{1, 2, 3\} = 3$;
- $\inf\{\frac{1}{n} \mid n \in \mathbb{N}\} = 0$;
- $\sup\{x \in \mathbb{Q} \mid 0 < x < 1\} = 1$;
- $\inf\{x \in \mathbb{Q} \mid 0 < x < 1\} = 0$;

We see that the supremum may or may not belong to the set and the same is true for the infimum.

A set A unbounded from above has $U(A) = \varnothing$ and hence it has no supremum. Dually, a set A unbounded from below has $L(A) = \varnothing$ and hence it has no infimum.

Does a set bounded from above has a supremum?

The answer to this question depends on the ambient linearly ordered set. For example, in $\mathbb{N}$ the answer is positive. Any subset of $\mathbb{N}$ bounded from above is finite, has the greatest element and the greatest element is the supremum.

On the other hand, if the supremum of A exists, but does not belong to A, then we can delete it from the whole set. For example, in $\mathbb{Q}$ consider $A = (0, 1)$, then $\sup A = 1$. Now remove 1 from $\mathbb{Q}$, that is consider $Y = \mathbb{Q} \smallsetminus \{1\}$. Then $A \subset Y$ is bounded from above, but has no supremum.

The latter example looks artificial. May it happen, say, in the set $\mathbb{Q}_{>0}$ of positive rational numbers without deleting the supremum from the ambient set?

Theorem 2. *The set $A = \{x \in \mathbb{Q}_{>0} \mid x^2 < 2\}$ is bounded from above, but has no supremum in $\mathbb{Q}_{>0}$.*