Isometries

1. Isometries versus moves

Isometries

In this course, the notion of move is initial and undefinable. The notion of congruent figures was introduced in terms of moves: two figures are called congruent if there exists a move mapping one of them to the other one. For some classes of figures there are easy congruence tests. For example, it is easy to check if two line segments are congruent.

One may ask whether a map of the plane to itself is a move if it maps figures of some type to congruent figures. A positive answer may give an additional insight on the notion of move.

In the context of this course, an isometry is a mapping of the plane to itself such that for any two points A, B the segment AB connecting them is congruent to the segment connecting their images.

The notion of *isometry* is a general notion commonly accepted in mathematics. The word isometry means "preserving distances". The word *metric* is a synonym to the word *distance*.

A move maps any figure to a congruent figure. In the definition of isometry this is required only for pairs of points, but for other figures this is not required. Therefore each move is an isometry.

Does the converse hold true? For some simple figures it is easy to prove that each isometry maps them to congruent figures.

1.A **Theorem.** *Any isometry f maps a circle c to a circle centered at the image of the center of c with radius congruent to the radius of c.*

Proof. Recall that a circle is the set of points X such that the segment OX is congruent to a fixed segment. So, $c = \{X \mid XO = AB\}$. An isometry f maps it to $\{f(X) \mid XO = AB\}$ that is to $\{f(X) \mid f(X)f(O) = AB\}$, because $f(X)f(O) = XO$. □

1.B **Theorem.** *Any isometry maps a segment to a congruent segment;*

Proof. Let us show that an isometry f maps a segment AB to segment $f(A)f(B)$. It is clear (from the definition of isometry) that the segment $f(A)f(B)$ congruent to AB. However, it is not clear that the image of the segment AB is a straight line segment. Points belonging to the segment AB are characterized as those points X for which the triangle inequality $AB \leq AX + XB$ turns to identity. In other words, X belongs to the segment AB if and only if $AX + XB = AB$. Since f is an isometry, $f(A)f(B) = AB$, $f(A)f(X) = AX$ and $f(X)f(B) = XB$ for any point X. Thus $f(A)f(B) = f(A)f(X) + f(X)f(B)$ if and only if $AX + XB = AB$. Therefore $f(X)$ belongs to the segment $f(A)f(B)$ if and only if X belongs to the segment AB. □

1 *Prove that any isometry maps*

(1) *a line to a line;*
(2) *a ray to a ray;*
(3) *a triangle to a congruent triangle;*
(4) *an angle to a congruent angle.*

In fact, each isometry of the plane is a move. We will prove this below by classifying all the isometries.

Recovering an isometry from images of three points

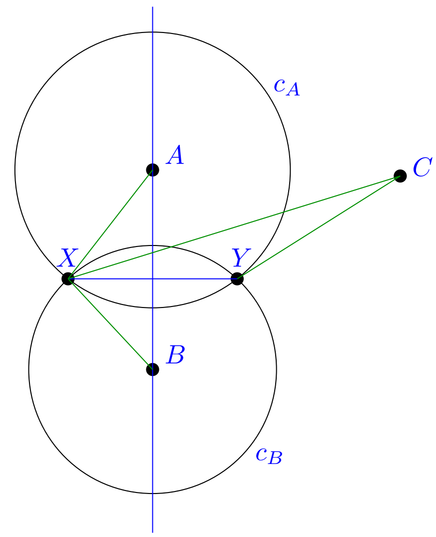
1.C **Theorem.** *An isometry of the plane can be recovered from its restriction to any triple of non-collinear points.*

Proof. Given images $f(A)$, $f(B)$ and $f(C)$ of non-collinear points A, B, C under an isometry f, let us find the image $f(X)$ of an arbitrary point X.

Draw circles c_A and c_B centered at A and B and passing through X. By theorem 1.A, the images $f(c_A)$ and $f(c_B)$ of these circles are circles of the same radii centered at $f(A)$ and $f(B)$. The circles c_A and c_B intersect in X. Therefore the circles $f(c_A)$ and $f(c_B)$ intersect in $f(X)$.

There may be at most one more intersection point. If $f(c_A)$ and $f(c_B)$ intersect in one point only, this point is $f(X)$, and we are done. If not, we have narrowed our search for $f(X)$ to two points, the two intersection points of $f(c_A)$ and $f(c_B)$, the images of intersection points X and Y of c_A and c_B.

Point C is not equidistant from X and Y, because the locus of equidistant points is the perpendicular to the segment XY bisecting it, points A, B are equidistant and belong to the locus, while C is not collinear with A and B. In order to choose the image of X, choose the intersection point of $f(c_A)$ and $f(c_B)$ such that the segment connecting it to $f(C)$ is congruent to XC.



□

In fact, as we will see later, there are exactly two isometries with the same restriction to a pair of distinct points.

Isometries as compositions of reflections

1.D **Theorem.** *Any isometry of the plane is a composition of at most three reflections.*

Proof. Choose three non-collinear points A, B, C. By theorem 1.C, it would suffice to find a composition of at most three reflections which maps A, B and C to their images under a given isometry f.

First, find a reflection R_1 which maps A to $f(A)$. The axis of such

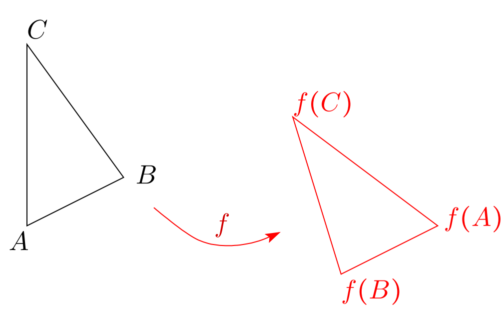


Figure 1.

a reflection is a perpendicular bisector of the segment $Af(A)$. It is uniquely defined, unless $f(A) = A$. If $f(A) = A$, one can take either a reflection about any line passing through A, or take, instead of reflection, an identity map for R_1 .

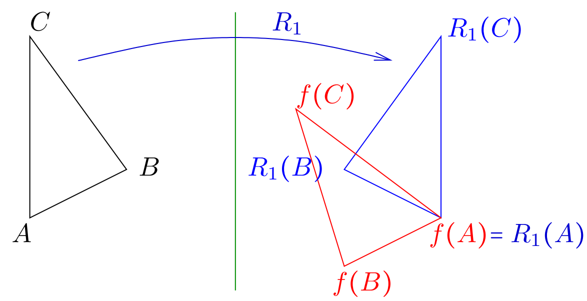


Figure 2.

Second, find a reflection R_2 which maps $R_1(B)$ to $f(B)$. The axis of such a reflection is the perpendicular bisector of the segment $R_1(B)f(B)$. The segments connecting $f(A)$ with $f(B)$ and $R_1(B)$ are congruent to the segment AB. Therefore $f(A)$ lies on the axis of reflection R_2 and is not moved by R_2.

Now either $R_2(R_1(C)) = C$ and then $S = R_2 \circ R_1$, or the triangles $R_2 \circ R_1(\triangle ABC)$ and $f(\triangle ABC)$ are symmetric in their common side $f(AB)$. In the former case $f = R_2 \circ R_1$, in the latter case denote by R_3

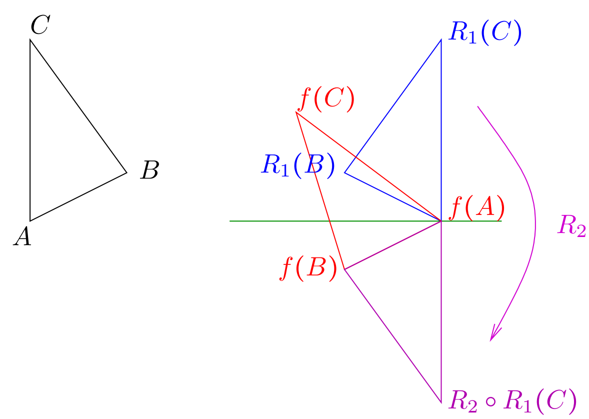
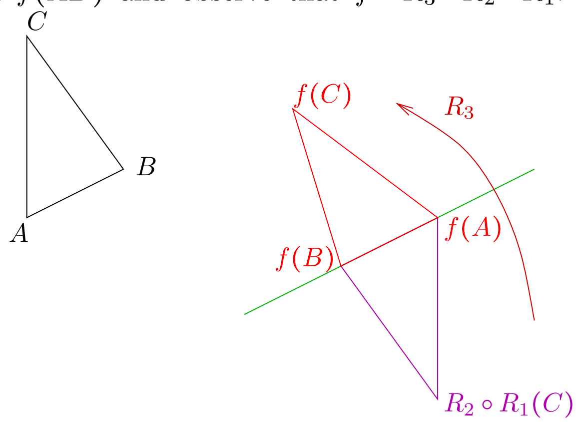


Figure 3.

the reflection in $f(AB)$ and observe that $f = R_3 \circ R_2 \circ R_1$.



□

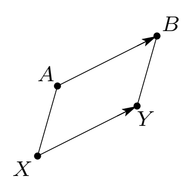
1.E **Corollary.** *Any isometry is a move. In particular, any isometry maps a figure to a congruent figure.*

Proof. By theorem 1.D, any isometry is a composition of reflections. Reflections are moves. A composition of moves is a move. □

Theorem 1.D gives an opportunity to investigate isometries by studying compositions of reflections. We will do this in the next sections.

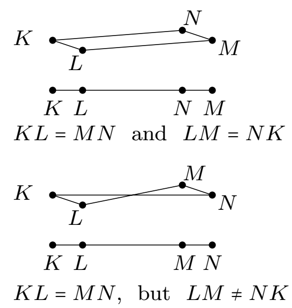
2. Translations

A map of the plane to itself is called a *translation* if, for some fixed points A and B, it maps a point X to a point $Y = T(X)$ such that $XYBA$ is a parallelogram.



Here we have to be careful with the notion of parallelogram, because a parallelogram may degenerate to a figure in a line. Not any degenerate quadrilateral fitting in a line deserves to be called a parallelogram, although any two sides of such a degenerate quadrilateral are parallel. In a (non-degenerate) parallelogram opposite sides are congruent and continuous degeneration cannot make them non-congruent. This motivates the following definition. By a parallelogram we mean a sequence of four segments KL, LM, MN and MK such that KL is *congruent and parallel* to MN and LM is *congruent and parallel* to MK. This definition describes both usual parallelograms, for which congruence of opposite sides can be deduced from parallelness and vice versa, and the degenerate parallelograms.

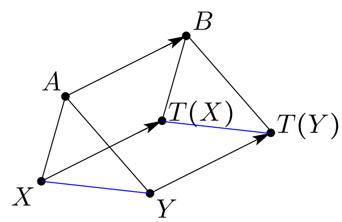
In fact, in this definition congruence of the sides from one of the pairs of opposite sides does not follow from congruence of the sides from the other pair.



2.A **Theorem.** *For any points A and B there exists a translation mapping A to B. Any translation is an isometry.*

Proof. Any three points A, B and X can be complemented in a unique way to a parallelogram $ABX'X$. Define $T(X) = X'$. Obviously, T satisfies the definition of translation and $T(A) = B$.

For any points X, Y, the quadrilateral $XYT(Y)T(X)$ is a parallelogram, since $XT(X) \| AB \| YT(Y)$ and $XT(X) = AB = YT(Y)$. Therefore, $XY = T(X)T(Y)$, so T is an isometry.



□

The translation moving a point A to a point B will be denoted below by $T_{\overrightarrow{AB}}$. The reflection in line l will be denoted by R_l.

Vectors

A pair of points A, B determines a segment AB. An ordered pair (A, B) of points determine an oriented segment. Orientation of a segment is nothing but an order of its end points. Usually in a picture an oriented segment is presented by an arrow and called an *arrow*, its first end point is called *arrow tail*, its second end point is called *arrowhead.* In formulas an arrow with tail A and head B is denoted by $\overrightarrow{AB}$.

Arrows $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are called *equivalent* if $ABDC$ is a parallelogram.

2 *Prove that this equivalence of vectors is an equivalence relation (i.e., it is reflective, symmetric and transitive). Compare to the proof of Theorem* 2.A.

An equivalence class of arrows is called a *vector.* A vector is determined by any of arrows belonging to it (i.e., by any element of the equivalence class).

An arrow $\overrightarrow{AB}$ defines the translation $T_{\overrightarrow{AB}}$ which maps A to B. Two arrows are equivalent if and only if they define the same translation.

Historical remarks. Vectors were introduced much later than other objects that we consider. They have been devised by William Rowan Hamilton He discovered quaternions (in 1843) as a generalization of complex numbers. In 1846 Hamilton divided his quaternions into the sum of real and imaginary parts that he respectively called "scalar" and "vector". The word vector comes from Latin and means carrier. Together with the term vector, Hamilton introduced a few other terms which did not survived. For example, he called the initial point of the arrow by the word 'vehend' (the one who is carried) and the final point (our arrowhead) by 'vectum' (the one who has been brought). Nowadays the term vector used in a huge number of meanings in many fields. Check with wikipedia.

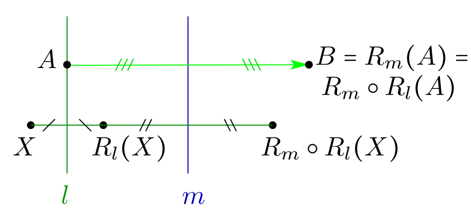
Reflections in parallel lines

2.B **Theorem.** *The composition of two reflections in parallel lines is a translation in the direction perpendicular to the lines, by twice the distance between the lines.*

More precisely, if lines l and m are parallel, and the line $A \in l$ and B is the image of A under the reflection in m, then

$$R_m \circ R_l = T_{\overrightarrow{AB}}.$$

Proof. Obviously, $R_l(A) = A$, as $A \in l$. Therefore $R_m \circ R_l(A) = R_m(A) = B$. Pick a point X near l, but outside the strip between l and m. The segment connecting X and $R_l(X)$ is bisected by l, the segment connecting $R_l(X)$ and $R_m \circ R_l(X)$ is bisected by m. Therefore the segment connecting X and $R_m \circ R_l(X)$ splits into four segments, the first two of them are congruent and the last two are congruent.



Thus the whole segment is congruent to the sum of its part enclosed between l and m and two segments outside the strip between l and m, and the whole segment is twice its part enclosed between l and m. Hence it is congruent to the segment connecting A and its image under $R_m \circ R_l$. Both the segments are perpendicular to l and m. Thus they are opposite sides of a parallelogram. We can take three such points close to each other and non-collinear. Restriction of the composition to them coincide with the translation $T_{\overrightarrow{AB}}$. Hence, by Theorem 1.C, these two isometries coincide. ∎

2.1. Non-uniqueness of presentations for a translation

Now we have two ways of presenting a translation: the original description via an arrow, and a presentation as a composition of reflections in two parallel lines. Both are not unique.

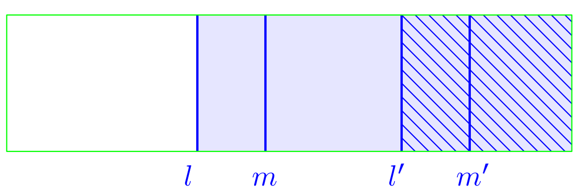
Non-uniqueness of the first one is well understood. Two arrows define the same translation if and only if they are equivalent, i.e., they represent the same vector. (Recall that arrows $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equivalent if $ABDC$ is a parallelogram.)

A presentation of a translation as a composition of reflections is quite similar to presentations via arrows. An arrow is an ordered pair of points. Reflections forming presentation of the second type are defined by their axes, which are parallel lines, and this pair of lines is ordered, because one of the reflection is performed first followed by the other one.

Which ordered pairs of parallel lines determine the same translation?

The answer to this question has been already obtained above, in Theorem 2.B. The lines should be perpendicular to the arrows that determine the translation, the distance between the lines is one half of the length of the arrows. These are the only restrictions on the pair of parallel lines.

2.C **Corollary of Theorem** 2.B. *If l, m a pair of parallel lines and l', m' is another pair of parallel lines, then $R_m \circ R_l = R_{m'} \circ R_{l'}$ if and only if lines l', m' are parallel to l and m, the distance between l' and m' equals the distance between l and m and the intersection of the half-plane bounded by l' and containing m' with the half-plane bounded by l and containing m is a half-plane.*

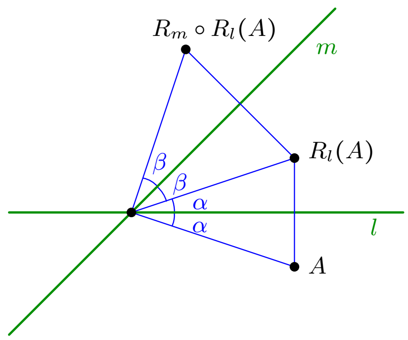


The following problem gives an elegant reformulation of 2.C .

3 *Let l, m, l', m' be lines, $l \| m$. Prove that $R_m \circ R_l = R_{m'} \circ R_{l'}$ if and only if there exists a translation T such that $l' = T(l)$ and $m' = T(m)$.*

3. Rotations

3.A **Theorem.** *The composition of two reflections in non-parallel lines is a rotation about the intersection point of the lines by the angle equal to doubled angle between the lines.*



Proof. Pick some points whose images under reflections are easy to track. From symmetries/congruent triangles in the picture, it is clear that effect of two refections is that of a rotation. Since we know that an isometry is determined by the image of 3 non-collinear points, there is no need to consider all possible positions of the points.

□

Non-uniqueness of presentation for a rotation

As for translations, a rotation can be presented as a composition of two reflections in many ways. However, the intersection point of the axes should be the center of the rotation. Therefore it should be the same for any two lines composition of reflections in which is the rotation. Similarly, the angle between the lines is the half of the angle of the rotation.

Conversely, any couple of lines passing through the center of a rotation, forming angle which equals the half of the rotation angle, provides a presentation of the rotation as a composition of two reflections. The reflections in the lines should be taken in the appropriate order, which depends on the direction of rotation.

4 *Let l, m, l', m' be lines, $l \cap m = O$. Prove that rotation $R_m \circ R_l$ coincides with $R_{m'} \circ R_{l'}$ if and only there exists a rotation R centered at O such that $R(l) = l'$ and $R(m) = m'$.*

4. Glide reflections

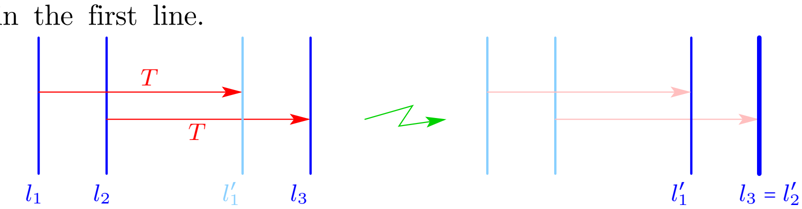
A reflection about a line l followed by a translation along l is called a *glide reflection.* In this definition, the order of reflection and translation does not matter, because they commute: $R_l \circ T_{\overrightarrow{AB}} = T_{\overrightarrow{AB}} \circ R_l$ if $l \parallel AB$. If $A = B$, then $T_{\overrightarrow{AB}}$ is identity and the glide reflection is a usual reflection.

Due to theorem **??**, $T_{\overrightarrow{AB}}$ can be presented as a composition of reflection in two parallel lines m and n perpendicular to l.

Compositions of three reflections

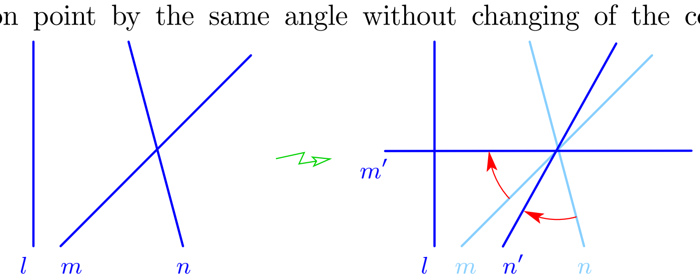
4.A **Theorem.** *A composition of three reflections is either a reflection, or a glide reflection.*

Proof. If all three axes of the reflections are parallel, then the firs two can be translated without changing of their composition (the composition of reflections about two parallel lines depends only on the direction of lines and the distance between them). By translating the first two lines, make the second of them coinciding with the third line. Then in the total composition they cancel, and the composition is just the reflection in the first line.

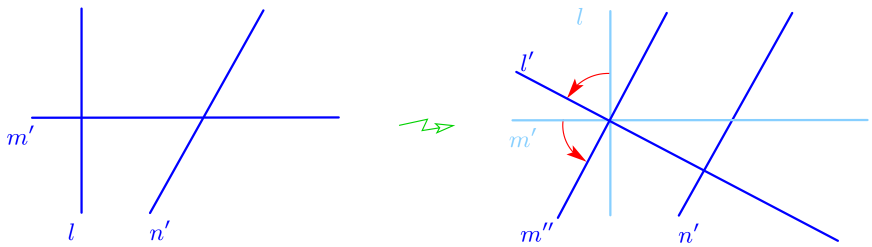


$$R_{l_3} \circ R_{l_2} \circ R_{l_1} = R_{l_3} \circ R_{l'_2} \circ R_{l'_1} = R_{l'_1}$$

If not all three lines are parallel, then the second is not parallel to one of the rest. The composition of reflections about these two non-parallel lines is a rotation, and the lines can be rotated simultaneously about their intersection point by the same angle without changing of the composition.



By an appropriate rotation, make the middle line perpendicular to the line which was not rotated. Then by rotating of these two perpendicular lines about their intersection point, make the middle one parallel to the other line. Now the configuration of lines consists of two parallel lines and a line perpendicular to them. The composition of reflections about them (the order does not matter any more, because they commute) is a glide reflection.



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5. Classification of plane isometries

5.A **Theorem.** *Any isometry of the plane is either the identity, or a reflection about a line, or rotation, or translation, or gliding reflection.*

Proof. By Theorem 1.D, any isometry of the plane is a composition of at most one reflections about lines. If the number of reflections in the composition is one, then the composition is a reflection. By Theorems 3.A and 2.B, a composition of two reflections is either rotation about a point, or translation. The identity isometry appears here, as the two lines may coincide (it appears also as the composition of zero reflections). By Theorem 4.A, a composition of three reflections is either a reflection or a glide reflection. □