MAT 310, Linear Algebra Name ________________________________

Homework 5

Score______________

1. Let V, W_1 and W_2 be finite-dimensional vector spaces and let $T_1 : V \to W_1$, $T_2 : V \to W_2$ be linear maps. Prove that there exists a linear map $S : W_1 \to W_2$ such that $T_2 = ST_1$ if and only if $\operatorname{null} T_1 \subset \operatorname{null} T_2$.

2. (a) For any vector space V over a field $\mathbb{F}$, construct an isomorphism $S_V : V \to \mathcal{L}(\mathbb{F}, V)$ (don't forget to prove that this is an isomorphism indeed).
(b) Find an isomorphism $\mathcal{L}(\mathbb{F}^n, V) \to V^n$.

3. Let U and V be subspaces of a vector space W. Prove that if $U + w = V$ for some vector $w \in W$, then $w \in U$ and $U = V$.

4. Following the proof of Theorem 4.8 from the textbook (page 121), prove the following Euclidean division theorem: For any positive integers p, s, there exist unique non-negative integers q, r such that
$p = sq + r$ and $r < s$. What objects in your proof would replace $\mathcal{P}_n(\mathbb{F})$ and its dimension?

5. Let V, U and W be finite-dimensional vector spaces over a field $\mathbb{F}$ and $p : V \to U$ and $q : V \to W$ be linear surjective maps such that $V = \operatorname{null} p \oplus \operatorname{null} q$.
(a) Prove that V is isomorphic to $U \times W$.
(b) Construct an isomorphism explicitly.
(c) Are the spaces V and $U \oplus W$ still isomorphic if they are not assumed to be finite-dimensional?