Span

Let V be a vector space, $U \subset V$ a subset.

Span

Let V be a vector space, $U \subset V$ a subset.

Definition. $\operatorname{span} U$ is the set of all linear combinations of $v_1, \ldots, v_m \in U$

Span

Let V be a vector space, $U \subset V$ a subset.

Definition. $\operatorname{span} U$ is the set of all linear combinations of $v_1, \ldots, v_m \in U$

(with any m).

Span

Let V be a vector space, $U \subset V$ a subset.

Definition. $\operatorname{span} U$ is the set of all linear combinations of $v_1, \ldots, v_m \in U$ (with any m).

Theorem. $\operatorname{span} U$ is the smallest subspace which contains U.

Span

Let V be a vector space, $U \subset V$ a subset.

Definition. $\operatorname{span} U$ is the set of all linear combinations of $v_1, \ldots, v_m \in U$ (with any m).

Theorem. $\operatorname{span} U$ is the smallest subspace which contains U.

What to prove?

Span

Let V be a vector space, $U \subset V$ a subset.

Definition. $\operatorname{span} U$ is the set of all linear combinations of $v_1, \ldots, v_m \in U$ (with any m).

Theorem. $\operatorname{span} U$ is the smallest subspace which contains U.

What to prove?

$\operatorname{span} U$ is a subspace.

Span

Let V be a vector space, $U \subset V$ a subset.

Definition. $\operatorname{span} U$ is the set of all linear combinations of $v_1, \ldots, v_m \in U$

(with any m).

Theorem. $\operatorname{span} U$ is the smallest subspace which contains U.

What to prove?

$\operatorname{span} U$ is a subspace.

$\operatorname{span} U$ is the smallest among subspaces.

Span

Let V be a vector space, $U \subset V$ a subset.

Definition. $\operatorname{span} U$ is the set of all linear combinations of $v_1, \ldots, v_m \in U$

(with any m).

Theorem. $\operatorname{span} U$ is the smallest subspace which contains U.

What to prove?

$\operatorname{span} U$ is a subspace.

$\operatorname{span} U$ is the smallest among subspaces.

Theorem. $\operatorname{span} U$ is the intersection of all subspaces which contain U.

Span

Let V be a vector space, $U \subset V$ a subset.

Definition. $\operatorname{span} U$ is the set of all linear combinations of $v_1, \ldots, v_m \in U$ (with any m).

Theorem. $\operatorname{span} U$ is the smallest subspace which contains U.

What to prove?

$\operatorname{span} U$ is a subspace.

$\operatorname{span} U$ is the smallest among subspaces.

Theorem. $\operatorname{span} U$ is the intersection of all subspaces which contain U.

Is the intersection of subspaces a subspace?

Linear dependence/independence

Let V be a vector space, $v_1, \ldots, v_m \in V$ be a list of vectors.

Linear dependence/independence

Let V be a vector space, $v_1, \dots, v_m \in V$ be a list of vectors.

Definition. $\operatorname{span}(v_1, \dots, v_m) = \{a_1 v_1 + \cdots + a_m v_m \mid a_1, \dots, a_m \in \mathbb{F}\}$

Linear dependence/independence

Let V be a vector space, $v_1, \ldots, v_m \in V$ be a list of vectors.

Definition. $\operatorname{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots + a_m v_m \mid a_1, \ldots, a_m \in \mathbb{F}\}$

Definition. A vector $v \in V$ **linearly depends** of $v_1, \ldots, v_m$ if $v \in \operatorname{span}(v_1, \ldots, v_m)$.

Linear dependence/independence

Let V be a vector space, $v_1, \dots, v_m \in V$ be a list of vectors.

Definition. $\operatorname{span}(v_1, \dots, v_m) = \{a_1 v_1 + \cdots + a_m v_m \mid a_1, \dots, a_m \in \mathbb{F}\}$

Definition. A vector $v \in V$ **linearly depends** of $v_1, \dots, v_m$ if $v \in \operatorname{span}(v_1, \dots, v_m)$.

$v_1, \dots, v_m \in V$ are called **linearly independent** if

Linear dependence/independence

Let V be a vector space, $v_1, \dots, v_m \in V$ be a list of vectors.

Definition. $\operatorname{span}(v_1, \dots, v_m) = \{a_1 v_1 + \cdots + a_m v_m \mid a_1, \dots, a_m \in \mathbb{F}\}$

Definition. A vector $v \in V$ **linearly depends** of $v_1, \dots, v_m$ if $v \in \operatorname{span}(v_1, \dots, v_m)$.

$v_1, \dots, v_m \in V$ are called **linearly independent** if none of $v_1, \dots, v_m$ linearly depends on the others.

Linear dependence/independence

Let V be a vector space, $v_1, \dots, v_m \in V$ be a list of vectors.

Definition. $\operatorname{span}(v_1, \dots, v_m) = \{a_1 v_1 + \dots + a_m v_m \mid a_1, \dots, a_m \in \mathbb{F}\}$

Definition. A vector $v \in V$ **linearly depends** of $v_1, \dots, v_m$ if $v \in \operatorname{span}(v_1, \dots, v_m)$.

$v_1, \dots, v_m \in V$ are called **linearly independent** if none of $v_1, \dots, v_m$ linearly depends on the others.

2.17 (More symmetric) **definition** List $v_1, \dots, v_m \in V$ is **linearly independent** if

$$a_1 v_1 + \dots + a_m v_m = 0 \quad \Longrightarrow \quad a_1 = \dots = a_m = 0 .$$

Linear Dependence Lemma

Linear Dependence Lemma

2.21 **Linear Dependence Lemma**

List $v_1, \ldots, v_m \in V$ is linearly dependent

Linear Dependence Lemma

2.21 Linear Dependence Lemma

List $v_1, \ldots, v_m \in V$ is linearly dependent $\iff$ $\exists j \in \{1, 2, \ldots, m\}$: $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$.

Linear Dependence Lemma

2.21 Linear Dependence Lemma

List $v_1, \ldots, v_m \in V$ is linearly dependent $\iff$ $\exists j \in \{1, 2, \ldots, m\}$: $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$.

List $v_1, \ldots, v_m \in V$ is linearly dependent $\iff$ $\exists$ a proper sublist $v_{k_1}, \ldots, v_{k_l}$ with the same span.

Independent $\leq$ spanning

In a finite-dimensional space,

Independent $\leq$ spanning

In a finite-dimensional space,

the length of every linearly independent list of vectors	$\leq$	the length of every spanning list of vectors

Independent $\leq$ spanning

In a finite-dimensional space,

the length of every linearly independent list of vectors	$\leq$	the length of every spanning list of vectors

Proof Let $u_1, \ldots, u_p$ is linearly independent in V ,

Independent $\leq$ spanning

In a finite-dimensional space,

the length of every linearly independent list of vectors	$\leq$	the length of every spanning list of vectors

Proof Let $u_1, \ldots, u_p$ is linearly independent in V, and $V = \operatorname{span}(w_1, \ldots, w_q)$.

Independent $\leq$ spanning

In a finite-dimensional space,

the length of every linearly independent list of vectors	$\leq$	the length of every spanning list of vectors

Proof Let $u_1, \ldots, u_p$ is linearly independent in V , and $V = \mathrm{span}(w_1, \ldots, w_q)$.
Prove: $p \leq q$.

Independent $\leq$ spanning

In a finite-dimensional space,

the length of every linearly independent list of vectors	$\leq$	the length of every spanning list of vectors

Proof Let $u_1, \ldots, u_p$ is linearly independent in V, and $V = \operatorname{span}(w_1, \ldots, w_q)$.
Prove: $p \leq q$. Plan:

Independent $\leq$ spanning

In a finite-dimensional space,

the length of every linearly independent list of vectors	$\leq$	the length of every spanning list of vectors

Proof Let $u_1, \ldots, u_p$ is linearly independent in V, and $V = \mathrm{span}(w_1, \ldots, w_q)$.
Prove: $p \leq q$. Plan: build up a list of length q

Independent $\leq$ spanning

In a finite-dimensional space,

the length of every linearly independent list of vectors	$\leq$	the length of every spanning list of vectors

Proof Let $u_1, \ldots, u_p$ is linearly independent in V, and $V = \operatorname{span}(w_1, \ldots, w_q)$.
Prove: $p \leq q$. Plan: build up a list of length q by gradual substituting w_i for u_j.

Independent $\leq$ spanning

In a finite-dimensional space,

the length of every linearly independent list of vectors	$\leq$	the length of every spanning list of vectors

Proof Let $u_1, \ldots, u_p$ is linearly independent in V , and $V = \text{span}(w_1, \ldots, w_q)$.
Prove: $p \leq q$. Plan: build up a list of length q by gradual substituting w_i for u_j .

$w_1, \ldots, w_q$

Independent $\leq$ spanning

In a finite-dimensional space,

the length of every linearly independent list of vectors	$\leq$	the length of every spanning list of vectors

Proof Let $u_1, \ldots, u_p$ is linearly independent in V, and $V = \mathrm{span}(w_1, \ldots, w_q)$.
Prove: $p \leq q$. Plan: build up a list of length q by gradual substituting w_i for u_j.

$w_1, \ldots, w_q \rightsquigarrow$

Independent $\leq$ spanning

In a finite-dimensional space,

the length of every linearly independent list of vectors $\leq$ the length of every spanning list of vectors

Proof Let $u_1, \ldots, u_p$ is linearly independent in V, and $V = \mathrm{span}(w_1, \ldots, w_q)$.
Prove: $p \leq q$. Plan: build up a list of length q by gradual substituting w_i for u_j.

$w_1, \ldots, w_q \rightsquigarrow u_1, w_1, \ldots, w_q$

Independent $\leq$ spanning

In a finite-dimensional space,

the length of every linearly independent list of vectors	$\leq$	the length of every spanning list of vectors

Proof Let $u_1, \ldots, u_p$ is linearly independent in V, and $V = \operatorname{span}(w_1, \ldots, w_q)$.
Prove: $p \leq q$. Plan: build up a list of length q by gradual substituting w_i for u_j.

$w_1, \ldots, w_q \rightsquigarrow u_1, w_1, \ldots, w_q$

$u_1, w_1, \ldots, w_q$ is linear dependent as $u_1 \in V = \operatorname{span}(w_1, \ldots, w_q)$

Independent $\leq$ spanning

In a finite-dimensional space,

the length of every linearly independent list of vectors	$\leq$	the length of every spanning list of vectors

Proof Let $u_1, \ldots, u_p$ is linearly independent in V, and $V = \text{span}(w_1, \ldots, w_q)$.
Prove: $p \leq q$. Plan: build up a list of length q by gradual substituting w_i for u_j.

$w_1, \ldots, w_q \rightsquigarrow u_1, w_1, \ldots, w_q$

$u_1, w_1, \ldots, w_q$ is linear dependent as $u_1 \in V = \text{span}(w_1, \ldots, w_q)$

$\exists j : w_j \in \text{span}(u_1, w_1, \ldots, w_{j-1})$ by 2.21

Independent $\leq$ spanning

In a finite-dimensional space,

the length of every linearly independent list of vectors	$\leq$	the length of every spanning list of vectors

Proof Let $u_1, \ldots, u_p$ is linearly independent in V , and $V = \operatorname{span}(w_1, \ldots, w_q)$.
Prove: $p \leq q$. Plan: build up a list of length q by gradual substituting w_i for u_j .

$w_1, \ldots, w_q \rightsquigarrow u_1, w_1, \ldots, w_q$

$u_1, w_1, \ldots, w_q$ is linear dependent as $u_1 \in V = \operatorname{span}(w_1, \ldots, w_q)$

$\exists j : w_j \in \operatorname{span}(u_1, w_1, \ldots, w_{j-1})$ by 2.21

Through w_j away from the list $u_1, w_1, \ldots, w_q$.

Independent $\leq$ spanning

In a finite-dimensional space,

the length of every linearly independent list of vectors	$\leq$	the length of every spanning list of vectors

Proof Let $u_1, \ldots, u_p$ is linearly independent in V, and $V = \text{span}(w_1, \ldots, w_q)$.
Prove: $p \leq q$. Plan: build up a list of length q by gradual substituting w_i for u_j.

$w_1, \ldots, w_q \rightsquigarrow u_1, w_1, \ldots, w_q$

$u_1, w_1, \ldots, w_q$ is linear dependent as $u_1 \in V = \text{span}(w_1, \ldots, w_q)$

$\exists j : w_j \in \text{span}(u_1, w_1, \ldots, w_{j-1})$ by 2.21

Through w_j away from the list $u_1, w_1, \ldots, w_q$. and continue.

■

Subspaces of a finite-dimensional space

2.26 A subspace of a finite-dimensional space is finite-dimensional.

Subspaces of a finite-dimensional space

2.26 A subspace of a finite-dimensional space is finite-dimensional.

Proof Let $U \subset V = \operatorname{span}(v_1, \ldots, v_p)$.

Subspaces of a finite-dimensional space

2.26 A subspace of a finite-dimensional space is finite-dimensional.

Proof Let $U \subset V = \operatorname{span}(v_1, \ldots, v_p)$.

Let us build a linear independent list $\subset U$.

Subspaces of a finite-dimensional space

2.26 A subspace of a finite-dimensional space is finite-dimensional.

Proof Let $U \subset V = \operatorname{span}(v_1, \ldots, v_p)$.

Let us build a linear independent list $\subset U$.

Lemma. If a list $w_1, \ldots, w_n \subset U$ is linear independent, but $U \neq \operatorname{span}(w_1, \ldots, w_n)$,

Subspaces of a finite-dimensional space

2.26 A subspace of a finite-dimensional space is finite-dimensional.

Proof Let $U \subset V = \mathrm{span}(v_1, \ldots, v_p)$.

Let us build a linear independent list $\subset U$.

Lemma. If a list $w_1, \ldots, w_n \subset U$ is linear independent, but $U \neq \mathrm{span}(w_1, \ldots, w_n)$, then $\exists w \in U$ such that $w_1, \ldots, w_n, w$ is linear independent.

Subspaces of a finite-dimensional space

2.26 A subspace of a finite-dimensional space is finite-dimensional.

Proof Let $U \subset V = \mathrm{span}(v_1, \ldots, v_p)$.

Let us build a linear independent list $\subset U$.

Lemma. If a list $w_1, \ldots, w_n \subset U$ is linear independent, but $U \neq \mathrm{span}(w_1, \ldots, w_n)$, then $\exists w \in U$ such that $w_1, \ldots, w_n, w$ is linear independent.

Reformulation. A linear independent list can be increased, unless it spans.

Subspaces of a finite-dimensional space

2.26 A subspace of a finite-dimensional space is finite-dimensional.

Proof Let $U \subset V = \operatorname{span}(v_1, \ldots, v_p)$.

Let us build a linear independent list $\subset U$.

Lemma. If a list $w_1, \ldots, w_n \subset U$ is linear independent, but $U \neq \operatorname{span}(w_1, \ldots, w_n)$, then $\exists w \in U$ such that $w_1, \ldots, w_n, w$ is linear independent.

■

Subspaces of a finite-dimensional space

2.26 A subspace of a finite-dimensional space is finite-dimensional.

Proof Let $U \subset V = \operatorname{span}(v_1, \ldots, v_p)$.

Let us build a linear independent list $\subset U$.

Lemma. If a list $w_1, \ldots, w_n \subset U$ is linear independent, but $U \neq \operatorname{span}(w_1, \ldots, w_n)$, then $\exists w \in U$ such that $w_1, \ldots, w_n, w$ is linear independent.

■

■

Subspaces of a finite-dimensional space

2.26 A subspace of a finite-dimensional space is finite-dimensional.

Proof Let $U \subset V = \mathrm{span}(v_1, \ldots, v_p)$.

Let us build a linear independent list $\subset U$.

Lemma. If a list $w_1, \ldots, w_n \subset U$ is linear independent, but $U \neq \mathrm{span}(w_1, \ldots, w_n)$, then $\exists w \in U$ such that $w_1, \ldots, w_n, w$ is linear independent.

Reformulation. A linear independent list can be increased, unless it spans.

Dual statement. A span of a vector space can be decreased, unless it is linearly independent.

Bases

2.27 **Definition** A **basis** of V is a list of vectors in V

Bases

2.27 **Definition** A **basis** of V is a list of vectors in V that is **linearly independent** and **spans** V.

Bases

2.27 **Definition** A **basis** of V is a list of vectors in V that is **linearly independent** and **spans** V.

2.28 **Examples**

- **Standard base** in $\mathbb{F}^n$: $(1, 0, \ldots, 0), (0, 1, 0 \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)$

Bases

2.27 **Definition** A **basis** of V is a list of vectors in V that is **linearly independent** and **spans** V.

2.28 **Examples**

- **Standard base** in $\mathbb{F}^n$: $(1,0,\ldots,0),(0,1,0\ldots,0),\ldots,(0,0,\ldots,0,1)$
- $1,z,\ldots,z^m$ is a basis in $\mathcal{P}_m(\mathbb{F})$.

Bases

2.27 **Definition** A **basis** of V is a list of vectors in V that is **linearly independent** and **spans** V.

2.28 **Examples**

- **Standard base** in $\mathbb{F}^n$: $(1, 0, \ldots, 0), (0, 1, 0 \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)$
- $1, z, \ldots, z^m$ is a basis in $\mathcal{P}_m(\mathbb{F})$.

2.29 **Criterion for basis**

Bases

2.27 **Definition** A **basis** of V is a list of vectors in V that is **linearly independent** and **spans** V.

2.28 **Examples**

- **Standard base** in $\mathbb{F}^n$: $(1,0,\ldots,0),(0,1,0\ldots,0),\ldots,(0,0,\ldots,0,1)$
- $1,z,\ldots,z^m$ is a basis in $\mathcal{P}_m(\mathbb{F})$.

2.29 **Criterion for basis**
$v_1,\ldots,v_n$ is a basis of V $\Longleftrightarrow$

Bases

2.27 **Definition** A **basis** of V is a list of vectors in V that is **linearly independent** and **spans** V.

2.28 **Examples**

- **Standard base** in $\mathbb{F}^n$: $(1, 0, \ldots, 0), (0, 1, 0 \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)$
- $1, z, \ldots, z^m$ is a basis in $\mathcal{P}_m(\mathbb{F})$.

2.29 **Criterion for basis**

$v_1, \ldots, v_n$ is a basis of V $\iff$ $\forall v \in V$

Bases

2.27 **Definition** A **basis** of V is a list of vectors in V that is **linearly independent** and **spans** V.

2.28 Examples

- **Standard base** in $\mathbb{F}^n$: $(1, 0, \ldots, 0), (0, 1, 0 \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)$
- $1, z, \ldots, z^m$ is a basis in $\mathcal{P}_m(\mathbb{F})$.

2.29 **Criterion for basis**

$v_1, \ldots, v_n$ is a basis of V $\iff$ $\forall v \in V \ \exists$ unique $a_1, \ldots, a_n \in \mathbb{F}$

Bases

2.27 **Definition** A **basis** of V is a list of vectors in V that is **linearly independent** and **spans** V .

2.28 Examples

- **Standard base** in $\mathbb{F}^n$: $(1,0,\ldots,0),(0,1,0\ldots,0),\ldots,(0,0,\ldots,0,1)$
- $1, z, \ldots, z^m$ is a basis in $\mathcal{P}_m(\mathbb{F})$.

2.29 **Criterion for basis**

$v_1,\ldots,v_n$ is a basis of V $\iff$ $\forall v \in V$ $\exists$ unique $a_1,\ldots,a_n \in \mathbb{F}$

$$v = a_1v_1 + \cdots + a_nv_n$$

Spanning list contains a basis

Spanning list contains a basis

2.31

Every spanning list in a vector space can be reduced to a basis of the vector space.

Spanning list contains a basis

2.31

Every spanning list in a vector space can be reduced to a basis of the vector space.

2.32 **Every finite-dimensional vector space has a basis.**

Linearly independent list extends to a basis

Linearly independent list extends to a basis

2.33

Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

Linearly independent list extends to a basis

2.33
Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

2.34 **Every subspace of V is part of a direct sum equal to V.**